Notes on torsion

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A "torsion" is a torsion of a linear connection. What is a linear connection? For this we need an n-dimensional manifold M (assumed here to be smooth). In the following we will take $n = 4$. We then have the *tangent bundle* TM. At a given point $p \in M$ the fiber T_pM of TM at p is the linear space of "tangent vectors" to M at p. A linear connection is a rule allowing us to "parallelly transport a tangent vector along a curve." There are many equivalent definitions (or "mathematical representations") of a linear connection (sometimes also called "affine connection"), the simplest one is given by: *to give a linear connection is the same as to give a covariant derivative.* Covariant derivative, usually denoted by the symbol ∇_X, applies to tensor fields (and also tensor densities, but that will not concern us here). On real (or complex) scalar functions on M it coincides with the usual derivative. It propagates to other tensor fields via the Leibnitz formula:

$$\nabla_X(UV) = (\nabla_X U)V + U(\nabla_X V) \tag{1}$$

and so it is enough to define it on vector fields. If x^μ is a local coordinate system, then $\Delta_X Y$ - the covariant derivative of the vector field $Y = Y^\mu \partial_\mu$ along the vector field $X = X^\mu \partial_\mu$ is given by

$$(\nabla_X Y)^\mu = \Gamma^\mu_{\nu\sigma}\, X^\nu Y^\sigma. \tag{2}$$

The connection ∇ is uniquely determined by $4 \times 4 \times 4 = 64$ functions ${\Gamma^\mu}_{\nu\sigma}(x)$ on M. Sometimes it is called *linear connection*, sometimes it is called *affine connection*, sometimes it is called *Cartan connection*, sometimes it is called *a connection associated to a principal connection in the bundle of linear frames of M*, sometimes we call it *a homogeneous part of the Poincaré group gauge field.* The term does not really matter. The formula – Eq.(2) – does.

Let us now do a simple exercise: we need a formula for a torsion of a connection that is uniquely determined by a "parallel frame". This kind of an exercise is useful for those contemplating *theories with an absolute parallelism* (recently: Gennady Shipov and Jack Sarfatti.)

In such a theory one usually starts with a frame e_a^μ, or a co–frame e_μ^a, satisfying

$$e_a^\mu e_\mu^b = \delta_a^b \tag{3}$$

and

$$e_a^\mu e_\nu^a = \delta_\nu^\mu . \tag{4}$$

The frame e_a^μ (and thus, due to the Leibnitz rule and the formula (3)) is assumed to be "parallel", that is we have

$$\nabla e_a = 0, \tag{5}$$

and

$$\nabla e^a = 0. \tag{6}$$

These two equations determine uniquely the connection ∇ and connection coefficients Γ in Eq. (2).

Every connection ∇, has two associated with it tensor fields: *curvature* and *torsion*. When ∇ is defined via the parallelism, as in Eqs (5),(6), the curvature is automatically zero, so the only interesting tensorial object is the torsion tensor. Torsion of any linear connection ∇ is defined via the formula [1]:

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{7}$$

The formula (7) above is not very explicit. It should be understood that it defines the torsion tensor $T_{\nu\sigma}^\mu$ via the equation

$$\tau(X, Y)^\mu = T_{\nu\sigma}^\mu X^\nu Y^\sigma . \tag{8}$$

Using the defining equations (5),(6) one can easily calculate the torsion tensor $T_{\nu\sigma}^\mu$ in terms of the tetrads. The result is:

$$T_{\nu\sigma}^\mu = -e_a^\mu (e_{\nu,\sigma}^a - e_{\sigma,\nu}^a) \tag{9}$$

where the comma denotes the partial derivative: $e_{\nu,\sigma}^a = \partial e_\nu^a / \partial x^\sigma$. It should be noted that there is no reason to call it "Cartan torsion" or "Ricci torsion" (or, for instance, Chanel torsion). It is just a torsion, as it is defined in any good textbook on differential geometry.

If our frame e_a, $a = 1, \ldots, n$ is supposed to be an orthonormal frame for the (contravariant) metric $g^{-1} = \eta^{ab} e_a \otimes e_b$, then, in addition to the teleparallel connection ∇, we also have the torsion–less Levi–Civita connection $\tilde{\nabla}$ of g. The difference is called the "contorsion tensor" $K^{\sigma}_{\mu\nu}$. So, by the very definition, for any vector fields X, Y we have

$$(\nabla_X Y)^k - (\tilde{\nabla}_X Y)^k = K^k_{lm} X^l Y^m. \tag{10}$$

Choosing $X = e_a, Y = e_b$, and renaming indices as necessary, we easily find that

$$K^c_{ab} = -(\tilde{\nabla}_a e_b)^c. \tag{11}$$

What Shipov calls "Ricci rotation" is nothing but the contorsion.

As defined above, torsion is a tensor field on the 4-dimensional manifold M. There is also another way of looking at the same object, namely as an object (vector valued two–form) defined on the bundle of frames, which is either $20 = 16 + 4$ dimensional - if we are dealing with $GL(4)$ gauge theory, or $10 = 6 + 4$ (or $11 = 6 + 1 + 4$ if we add dilations) - when we reduce the structure group to the Lorentz group. In such a theory we have a "soldering form", and its exterior covariant derivative - both defined on the bundle of frames. This is explained, for instance, in [2].

Update, October 8, 2010
This morning I received an email containing, in particular, these words: "*I am already familiar with the Cartan's structural equations and the problem is, I am not able to calculate T and R when one of R or T is not equal to zero.*" Therefore I am updating a little bit this file.

Eq. (7) is valid with and without curvature. When we have curvature, then we have to decide whether we want to calculate in a coordinate frame (so called *holonomic* or in an orthonormal frame *non-holonomic*. Let us start with the orthonormal frame e_a and its dual co-frame e^a. By definition we have $e^a_\mu e^\nu_a = \delta^\nu_\mu$. In the case of non-zero curvature it is impossible to choose an orthonormal frame that is parallel. When we calculate $\nabla_X e_a$, we obtain another vector field that can be decomposed with respect to the basis e_a :

$$\nabla_X e_b = \Gamma^c_{X,b} e_c,$$

where $\Gamma^c_{X,b}$ are the coefficient functions that depend linearly on the vector field X. Suppose we take $X = e_a$, and we denote the corresponding $\Gamma^c_{X,b}$ by

Γ^c_{ab}. Then we have

$$\nabla_a e_b = \Gamma^c_{ab} e_c.$$

The coefficients Γ^c_{ab} characterize the connection. A priori they can be arbitrary, but if our connection preserves the metric tensor and if our frame e_a is orthonormal, then, for each a the matrix $(\Gamma_a)^c_b$ must be in the Lie algebra of the Lorentz group. Choosing $X = e_a, Y = e_b$ we write Eq. (7) as

$$\tau(e_a, e_b) = \nabla_a e_b - \nabla_b e_a - [e_a, e_b].$$

Now $\tau(e_a, e_b)$ is a vector field that can be decomposed:

$$\tau(e_a, e_b) = T^c_{ab} e_c.$$

The functions T^c_{ab} are torsion coefficients calculated with respect to the frame e_a. So, we have:

$$T^c_{ab} = (\nabla_a e_b)^c - (\nabla_b e_a)^c - [e_a, e_b]^c.$$

The first two terms are now expressed through the connection coefficients, the commutator can be calculated much as in the case of the zero curvature. The result is:

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba} - e^\mu_a e^\nu_b (e^c_{\mu,\nu} - e^c_{\nu,\mu}).$$

On the other hand, if we want to calculate torsion coefficients in a holonomic frame $\partial_\mu = \frac{\partial}{\partial x^\mu}$, then the connection coefficients are given by

$$\nabla_\mu \partial_\nu = \Gamma^\sigma_{\mu\nu}\, \partial_\sigma.$$

The commutator term in the definition of torsion vanishes, because our frame is holonomic (partial derivatives commute): $[\partial_\mu, \partial_\nu] = 0$, therefore we get the simple expression:

$$T^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}.$$

As for the curvature, generally defined by the formula:

$$\nabla_X \nabla_Y\, Z - \nabla_Y \nabla_X\, Z - \nabla_{[X,Y]} Z = R_{X,Y}\, Z,$$

in an orthonormal frame, it is given by a simple formula:

$$\nabla_a \nabla_b\, e_c - \nabla_b \nabla_a\, e_c - \nabla_{[e_a,e_b]}\, e_c = R_{ab}{}^c{}_d\, e_d,$$

where

$$R_{ab} = \partial_a \Gamma_b - \partial_b \Gamma_a + [\Gamma_a, \Gamma_b] - f^c_{ab} \Gamma_c.$$

Here Γ_a stands for the matrix Γ^c_{ab}, and f^c_{ab} are the *structure functions* for the frame e_a :

$$[e_a, e_b] = f^c_{ab}\, e_c.$$

The conventions for denoting the curvature coefficients differ. Some authors define it with a minus sign, some other write R^c_{dab} for the above $R_{ab}{}^c{}_d$. With our notation, explicitly:

$$R_{ab}{}^c{}_d = \partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad} + (\Gamma^c_{am}, \Gamma^m_{bd} - \Gamma^c_{bm}, \Gamma^m_{ad}) - f^m_{ab} \Gamma^c_{md}.$$

1 Note on Gennady Shipov's by Jack Sarfatti

The first sentence in [3] reads:

> In the tetrad notation, the Einstein equivalence principle (EEP) *local passive map* from the curved $g_{\mu\nu}$ LNIF space–time to the flat η_{ij} LIF tangent vector fiber space at a fixed point event is
>
> $$g_{\mu\nu} = e^i_\mu \eta_{ij} e^j_\nu. \qquad (12)$$

We have here the following undefined terms "local passive map", "curved $g_{\mu\nu}$ LNIF space–time", "flat η_{ij} LIF tangent vector fiber space", "fixed point event". Otherwise the formula above is well known: it relates metric coefficients in a local coordinate system to a tetrad of four orthonormal vectors. All the words in quotation marks above are unnecessary or misleading, or make no sense at all. Their actual function is to discourage the experts from trying to figure it out what the author is talking about. It is part of what is called "impressionistic style" in theoretical physics. There is nothing wrong with impressionistic style. Some painters are realist, some surrealist, some impressionist etc. But it is important to recognize the style. When I see an impressionistic painting, I usually squint my eyes so as to consciously not to pay attention to the details. I understand that it is up to me to give the meaning to the painting, not to the painter. And sometimes I am able to give this meaning, and sometimes not.

to be continued...

2 Acknowledgements

Thanks are due to R.O. for reading these notes and for pointing out a bunch of typos.

References

[1] Theodore Frankel, *The geometry of physics*, Cambridge University Press, 1997, Eq. 9.14. The formula (9.14) there has a missprint, it has $\nabla_Y X - \nabla_X Y$ instead of $\nabla_X Y - \nabla_Y X$

[2] A. Jadczyk, *Vanishing Vierbein*, Eq. (14) therein, see: http://quantumfuture.net/quantum_future/papers/vv/index.html and/or http://arxiv.org/abs/gr-qc/9909060

[3] Jack Sarfatti, *Shipov's Torsion Field. Note on Gennady Shipov's by Jack Sarfatti. 6/5/04, 11:41 AM*,