Notes on torsion

Arkadiusz Jadczyk

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A "torsion" is a torsion of a linear connection. What is a linear connection? For this we need an n-dimensional manifold M (assumed here to be smooth). In the following we will take $n = 4$. We then have the *tangent bundle* TM. At a given point $p \in M$ the fiber T_pM of TM at p is the linear space of "tangent vectors" to M at p. A linear connection is a rule allowing us to "parallelly transport a tangent vector along a curve." There are many equivalent definitions (or "mathematical representations") of a linear connection (sometimes also called "affine connection"), the simplest one is given by: *to give a linear connection is the same as to give a covariant derivative.* Covariant derivative, usually denoted by the symbol ∇) applies to tensor fields (and also tensor densities, but that will not concern us here). On real (or complex) scalar functions on M it coincides with the usual derivative. It propagates to other tensor fields via the Leibnitz formula:

$$\nabla_X(UV) = (\nabla_X U)V + U(\nabla_X V) \tag{1}$$

and so it is enough to define it on vector fields. If x^μ is a local coordinate system, then $\Delta_X Y$ - the covariant derivative of the vector field $Y = Y^\mu \partial_\mu$ along the vector field $X = X^\mu \partial_\mu$ is given by

$$(\nabla_X Y)^\mu = \Gamma^\mu_{\nu\sigma}\, X^\nu Y^\sigma. \tag{2}$$

The connection ∇ is uniquely determined by $4 \times 4 \times 4 = 64$ functions ${\Gamma^\mu}_{\nu\sigma}(x)$ on M. Sometimes it is called *linear connection*, sometimes it is called *affine connection*, sometimes it is called *Cartan connection*, sometimes it is called *a connection associated to a principal connection in the bundle of linear frames of M*, sometimes we call it *a homogeneous part of the Poincaré group gauge field.* The term does not really matter. The formula – Eq.(2) – does.

Let us now do a simple exercise: we need a formula for a torsion of a connection that is uniquely determined by a “parallel frame”. This kind of an exercise is useful for those contemplating *theories with an absolute parallelism* (recently: Gennady Shipov and Jack Sarfatti.)

In such a theory one usually starts with a frame e^μ_a, or a co–frame e^a_μ, satisfying

$$e^\mu_a e^b_\mu = \delta^b_a \tag{3}$$

and

$$e^\mu_a e^a_\nu = \delta^\mu_\nu. \tag{4}$$

The frame e^μ_a (and thus, due to the Leibnitz rule and the formula (3)) is assumed to be “parallel”, that is we have

$$\nabla e_a = 0, \tag{5}$$

and

$$\nabla e^a = 0. \tag{6}$$

These two equations determine uniquely the connection ∇ and connection coefficients Γ in Eq. (2).

Every connection ∇, has two associated with it tensor fields: *curvature* and *torsion*. When ∇ is defined via the parallelism, as in Eqs (5),(6), the curvature is automatically zero, so the only interesting tensorial object is the torsion tensor. Torsion of any linear connection ∇ is defined via the formula [1]:

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{7}$$

The formula (7) above is not very explicit. It should be understood that it defines the torsion tensor $T^\mu_{\nu\sigma}$ via the equation

$$\tau(X,Y)^\mu = T^\mu_{\nu\sigma} X^\nu Y^\sigma. \tag{8}$$

Using the defining equations (5),(6) one can easily calculate the torsion tensor $T^\mu_{\nu\sigma}$ in terms of the tetrads. The result is:

$$T^\nu_{\nu\sigma} = -e^\mu_a (e^a_{\nu,\sigma} - e^a_{\sigma,\nu}) \tag{9}$$

where the comma denotes the partial derivative: $e^a_{\nu,\sigma} = \partial e^a_\nu / \partial x^\sigma$. It should be noted that there is no reason to call it ”Cartan torsion” or ”Ricci torsion” (or, for instance, Chanel torsion). It is just a torsion, as it is defined in any good textbook on differential geometry.

As defined above, torsion is a tensor field on the 4-dimensional manifold M. There is also another way of looking at the same object, namely as an object (vector valued two–form) defined on the bundle of frames, which is either $20 = 16 + 4$ dimensional - if we are dealing with $GL(4)$ gauge theory, or $10 = 6 + 4$ (or $11 = 6 + 1 + 4$ if we add dilations) - when we reduce the structure group to the Lorentz group. In such a theory we have a "soldering form", and its exterior covariant derivative - both defined on the bundle of frames. This is explained, for instance, in [2].

There is also the third way of interpreting the formula (9), namely in terms of second–order frames.

to be continued...

References

[1] Theodore Frankel, *The geometry of physics*, Cambridge University Press, 1997, Eq. 9.14. The formula (9.14) there has a missprint, it has $\nabla_Y X - \nabla_X Y$ instead of $\nabla_X Y - \nabla_Y X$

[2] A. Jadczyk, *Vanishing Vierbein*, Eq. (14) therein, see: http://www.cassiopaea.org/quantum_future/papers/vv/index.html and/or http://arxiv.org/abs/gr-qc/9909060