1 Introduction

Dear Bill,

So, you have chosen Flanders as the basic text . Thanks for scanning these two sections for me - I have ordered the book from Amazon, but it will not arrive before Christmas. So, you have chosen Flanders, and I promised that I will accept any of your choices, provided it is sufficiently precise. We got a problem here. Flanders is NOT sufficiently precise. It is not a mathematical textbook. Notice, we have sections 8.3 and 8.4, which are gonna be basic for us, but there is no one "Definition" or "Lemma" or "Theorem" or "Corollary" there! This is not a mathematical book. This book talks about mathematics, but it is not mathematics. For instance, look at p. 127. WE have the symbol **dP** Flanders then comments: "Since **dP** is in no sense an exterior derivative of anything, we distinguish this **d** by bold-face type from the usual d", but he would not tell you what this symbol **dP** is! We will find more places like that. Taking Flanders as the basic text will not be easy. Yet, I believe, I can deal with it. It will just take several "lessons", so be patient, please. We have to go step by step, feeling in the holes. To simplify notation, I will use regular font type rather than bold as in the book, I will also use upper case D for his d. Let us start with

> "8.5. Affine Connection". He writes: "We now want to associate with each vector field v on M a vector field Dv with one-form coefficients. We must be able to do this for the vector fields e_i of the basis, so we require
>
> $$De_i = \sum \omega_i^{\ j} e_j \tag{1}$$
>
> where $\omega_i^{\ j}$ are one-forms on the neighborhood U.

But then he adds:

> "There are certain consistency relations which guarantee that the computation of Dv will be independent of any frame."

What are these consistency relations is explained only later on. Now we have a "definition":

> Locally we may describe an affine connection as follows. We are given U, the affine frame $e_1, \ldots, e_n$, and the dual basis $\sigma^1, \ldots, \sigma^n$ of one-forms. An affine connection consists of n^2 on-forms $\omega_i^{\ j}$ subject to no constrains at all.

So, here we are. What we are lacking are only these "certain consistency relations." So let us not hesitate to bring them in, so that we have a complete definition. Consistency relations are discussed at the bottom of p. 145 and continue to p. 146. The problem is this: what if we have, in a given neighborhood, two frames, e and $\bar{e}$, how the respective connection forms ω and $\bar{\omega}$ should be related to assure that Dv is the same - whether computed using one frame, or another. These consistency relations are written as

$$\bar{\Omega} = A\Omega A^{-1} + (dA)A^{-1}, \tag{2}$$

where

$$\bar{e} = Ae. \tag{3}$$

We will have to expand these formulae to make them explicit. Here Ω is the matrix of ${\omega_i}^j$ and A is also a matrix. Writing explicitly:

$$\bar{e}_i = {A_i}^j e_j \tag{4}$$

and

$${\bar{\omega}_i}^j = {A_i}^k {\omega_k}^l {(A^{-1})_l}^j + {(dA)_i}^l {(A^{-1})_l}^j \tag{5}$$

So, now we are done. Now we know how to transport parallelly vector along a curve. Suppose we have vector v along a curve. Choose arbitrarily frame e (it must be defined along the curve). Let ${\omega_i}^j$ be connections one-forms with respect to this frame. Decompose vector v with respect to basic vectors: $v = \sum f^i e_i$ then the condition for parallel transport is

$$df^j + \sum f^i \cdot {\omega_i}^j = 0 \tag{6}$$

calculated on vector tangent to the curve.
We are done with definitions. It is important to notice that nowhere above we had to use coordinates on the manifold, nowhere we had to write components of our differential forms. We can do this too, as Flanders does at the top of p.144, when he says "We shall quickly point out the relation to the customary formulation of an affine connection". And we will return to this point later on. But, for a while, notice that Flanders is aware of the fact that his formulation is not a customary one. We will have to bear this in mind in order not to mix the two formulations. Taking a mathematical formula from one set of conventions and using it, literally, in another set of conventions,

can easily result in an error. And, in fact, it does lead to a confusion in our particular case of analyzing Shipov's Chapter 5.4.
To summarize: to give a connection is to give n^2 differential forms $\omega_i^{\ j}$ and a frame e_i. Then we call $\omega_i^{\ j}$ connection form (matrix valued) with respect to the frame e_i. If someone prefers to choose another frame $\bar{e}_i$, but wants to work with the same connection as we do, then he must use another set of forms $\bar{\omega}_i^{\ j}$, related to $\omega_i^{\ j}$ by Eq. (5) If someone gives us just n^2 one-forms $\omega_i^{\ j}$, but does not tell us his frame - then we do not know his connection. If somebody gives us n^2 one-forms $\omega_i^{\ j}$, and gives us two different frames, then we do not know what the connection is until he specifies with respect to WHICH of his two frames our forms define a connection. Example: I have two frames, e_i and $\bar{e}_i$. But I am not going to tell you which one is MY frame. Yet I will tell you my n^2 connection forms: THEY ARE ALL ZERO! Legitimate choice. isn't it! Look now at Eq. (6). If all n^2 one-forms $\omega_i^{\ j}$ are zero, then the condition for parallel transport is that components f^i of a vector v are *constant* during the parallel transport. But components of vector v with respect to which frame? With respect to e_i or with respect $\bar{e}_i$? I did not tell you which is MY frame! If e_i is my frame, then connection forms being zero means that vectors of this frame are parallel (as components of frame vectors with respect to itself are constant - Kronecker deltas). But if $\bar{e}_i$ is MY frame, then $\bar{e}_i$ are parallel. And if $\bar{e}_i$ relates to e_i by a non-constant matrix A, then both statements: $\bar{e}_i$ is parallel AND e_i cannot be true at the same time. You can see it also from equation (5): suppose that matrix A is non-constant, then, if $\omega_i^{\ j} = 0$ then $\bar{\omega}_i^{\ j}$ is non-zero, and if $\bar{\omega}_i^{\ j} = 0$ then $\omega_i^{\ j}$ is non-zero.
That was in order to illustrate very important fact: to define a connection is to give n^2 forms $\omega_i^{\ j}$ AND to specify a frame.

2 Necessary changes in notation

We want to move now to Shipov's equation (5.67). We will analyze it, we will try to understand what it says. So, here it is:

$$\Delta^a_{\ b} = e^a_{\ i} de^i_{\ b} \tag{7}$$

Now, to minimize possible confusion, I will follow your convention of using upper case letter for frame vectors. Thus I will write

$$\Delta^a_{\ b} = e^a_{\ i} dE^i_{\ b} \tag{8}$$

Let us analyze now, what do we have? On the left hand side we have something that looks like n^2 one-forms $\Delta^a_{\ b}$. Our guess is that they must somehow correspond to what Flanders called ω_i^j. First we notice that Shipov uses letters a, b, c, d while Flanders uses letters i, j, k, l. This is not a big deal for us. Everywhere in Flanders' formulas we will replace i, j, k, l by a, b, c, d and then we can use Flanders's formulas. But then we have another difference, namely Shipov writes $\Delta^a_{\ b}$ while Flanders writes *omab*. Notice the oder of indices is different. Shipov is using transpose of Flanders' formulas. Trying to relate one to another, the best thing is to rewrite Flanders formulas using Shipov's convention. So, let us do it now. Transposing Eq. (1) becomes:

$$De_a = \sum e_b \omega^b_{\ a} \tag{9}$$

Eq. (4) transposes to:

$$\bar{e}_a = e_b A^b_{\ a} \tag{10}$$

while Eq. (5) transposes to

$$\bar{\omega}^a_{\ b} = (A^{-1})^a_{\ c} \omega^c_{\ d} A^d_{\ b} + (A^{-1})^a_{\ c} d(A^c_{\ b}) \tag{11}$$

Parallel transport formula (6) transposes to

$$df^a + \omega^a_{\ b} f^b = 0. \tag{12}$$

That all this is correct can be also checked by explicit calculations, repeating the steps in Flanders, but using the new conventions.
Now we are ready to attack Shipov (5.67)

3 Shipov's puzzle (5.67)

After all this preparation we are ready now to interpret Shipov's formula (5.67) ... or, are we? What strikes when we contemplate formula (8) is that we have now two kind of indices: a and b that corresponds to Flanders i and j, but also we have another index: summation index i in (8). We know, from earlier sections of Shipov, that while Latin index $a, b, ...$ refers to vierbein vectors and dual forms E_a and e^a respectively, letters $i, j, k, ...$ refer to a coordinate system x^i. Therefore, apparently, in order to define n^2 forms $\Delta^a_{\ b}$ in Eq. (8) we need *two objects:* moving frame E_a AND a coordinate system x^i. Otherwise we can not even define our connection form. Let us first ask: which frame E_a we have in mind? This is explained in previous section. We assume, by the very nature of distant parallelism, that there is a preferred frame (or *vierbein*) that sets a standard for parallelism. It is enough to fix vectors E_a at one point - and this fixes them all over the space! The only freedom left is that of a rotation by a constant, orthogonal, matrix, and this is so trivial, that we are not even going to discuss it. This settles the question: which E_a to chose in Eq. (8). But it does not settles the question: which x^i to chose? So, we got a problem. But this is not the only problem, You may think now that my question is silly, because there is an evident "answer", namely that *any* coordinate system will do, that it is irrelevant which one to use, because all will define the same connection. You will soon see that this optimism is NOT justified. Anyway, before even trying to address the problem, let us list our first question:

Question 1 Which coordinate system should be used in the definition (8)?

Now, let us suppose that a coordinate system x^i has been specified. Then Eq. (8) indeed defines n^2 differential forms. Therefore, according to our previous discussion of affine connections, we do have a connection! Or, do we? Didn't we stressed that in order to have a connection we need two objects: n^2 one-forms AND a frame. But now we have n^2 one-forms and TWO frames! One is our vierbein E_a, but we also have the holonomic frame ∂_i consisting of vectors tangent to coordinate lines $\partial_i = \frac{\partial}{\partial x^i}$. Of course we expect that, if we were supposed to refer our n^2 forms to the holonomic frame ∂_i, then probably we will have to convert indices in *deab* from a, b to i, j. Therefore we have the following alternative:

Question 2 Assuming that a coordinate system x^i has been chosen, which of the following two interpretations of Eq. (8) serving as definition of a connection should we take:

1. Forms $\Delta^a{}_b$ define our connection with respect to the frame E_a, the same frame that enters Eq. (8)

2. Forms $\Delta^i{}_j = E^i{}_a \Delta^a{}_b e^b{}_j$ define our connection with respect to the holonomic frame $\partial_i = \frac{\partial}{\partial x^i}$, frame which is implicitly assumed in Eq. (8)

Of course, being optimists and benevolent and open minded, we hope that our world is best of possible worlds, and therefore our two questions will be answered as follows: "it does not matter which x^i we take, all define the same connection, this being true whether we take option 1 or option 2 above.

As we will see, our world is not the best of all possible worlds. In fact, we will see that, apparently, our world, if governed by Shipov's formula (8), is the worst of all possible worlds.

3.1 Does a coordinate system matter?

Let us consider first the hypothesis 1. That is we expect that the forms $\Delta^a_{\ b}$ are connection forms with respect to our frame E_a. We are not going to change this frame, we like it, why should we rotate it? Finally it is the frame that defines distant parallelism of the previous Sections 5.2 and 5.3 of Shipov. Because we are not going to change our frame, therefore formula (11) does not concern us. Also, because forms $\Delta^a_{\ b}$ are supposed to be forms of a connection with respect to E_a (and NOT ∂_i - which will be our hypothesis 2.) - they should stay the same if we change coordinate system x^i and replace it by another coordinate system, say $\bar{x}^i$. Equation (8), we hope, is such a nice equation, that it defines the same forms $\Delta^a_{\ b}$ independently of which coordinate system we use to compute these forms. We know that such things do happen. For instance we have:

$$E^i_{\ a}e^a_{\ j} = \delta^i_j \tag{13}$$

and this holds, as we know, in any coordinate system! How do we know it? Okay, let us do this simple exercise, just to warm up before testing our hypothesis 1. First, notice that we have, by definition,

$$E_a = \partial_i \cdot E^i_{\ a} \tag{14}$$

and

$$e^a = e^a_{\ i} dx^i \tag{15}$$

Notice that in (14) we are not differentiating $E^i_{\ a}$ (that is why we have put the "dot" there, we are simply keeping to our convection of acting with matrices from the right, as in (10). We know that the coefficients $E^i_{\ a}$ and $e^a_{\ i}$ can be also expressed by evaluation of differential forms:

$$E^i_{\ a} = dx^i(E_a), \tag{16}$$

$$e^a_{\ i} = e^a(\partial_i), \tag{17}$$

as can be easily checked by applying dx^i to Eq. (14) and applying Eq. (15) to ∂_i. Now, suppose, we have also another coordinate system, which we will denote as $x^{i\prime}$. The we get:

$$E^{i\prime}{}_{a} = dx^{i\prime}(E_a) = \frac{\partial x^{i\prime}}{\partial x^i} dx^i(E_a) = \frac{\partial x^{i\prime}}{\partial x^i} E^i{}_a, \tag{18}$$

or

$$E^{i\prime}{}_{a} = \frac{\partial x^{i\prime}}{\partial x^i} E^i{}_a \tag{19}$$

and similarly

$$e^a{}_{i\prime} = \frac{\partial x^i}{\partial x^{i\prime}} e^a{}_i \tag{20}$$

Therefore we get:

$$E^{i\prime}{}_{a} e^a{}_{j\prime} = \frac{\partial x^{i\prime}}{\partial x^i} E^i{}_a \frac{\partial x^j}{\partial x^{j\prime}} e^a{}_j = \frac{\partial x^{i\prime}}{\partial x^i} \frac{\partial x^j}{\partial x^{j\prime}} \delta^i_j = \frac{\partial x^{i\prime}}{\partial x^i} \frac{\partial x^i}{\partial x^{j\prime}} = \delta^{i\prime}_{j\prime} \tag{21}$$

as expected.

Now, consider Eq. (8). We want to compute our n^2 connection forms using the primed coordinate system. Since we do not know, a priori, whether the result will be independent of the chosen coordinate system, let us denote these forms as Δ':

$$\Delta'^a{}_b = e^a{}_{i\prime} dE^{i\prime}{}_b \tag{22}$$

If we are lucky, the we will find that $\Delta'^a{}_b = \Delta^a_b$ Substituting (19) and (20) into (22) we get

$$\Delta'^a{}_b = \frac{\partial x^i}{\partial x^{i\prime}} e^a{}_i d(\frac{\partial x^{i\prime}}{\partial x^j} E^j{}_b) = \frac{\partial x^i}{\partial x^{i\prime}} e^a{}_i d(\frac{\partial x^{i\prime}}{\partial x^j}) E^j{}_b + \frac{\partial x^i}{\partial x^{i\prime}} e^a{}_i \frac{\partial x^{i\prime}}{\partial x^j} d(E^j{}_b) \tag{23}$$

thus

$$\Delta'^a{}_b = \frac{\partial x^i}{\partial x^{i\prime}} d(\frac{\partial x^{i\prime}}{\partial x^j}) e^a{}_i E^j{}_b + e^a{}_i \delta^i_j d(E^j{}_b) = \frac{\partial x^i}{\partial x^{i\prime}} d(\frac{\partial x^{i\prime}}{\partial x^i}) e^a{}_i E^j{}_b + e^a{}_i d(E^i{}_a) \tag{24}$$

or

$$\Delta'^a{}_b = \Delta^a{}_b + \frac{\partial x^i}{\partial x^{i\prime}} d(\frac{\partial x^{i\prime}}{\partial x^i}) e^a{}_i E^j{}_b \tag{25}$$

We see that we got the second term, and there is no way to make this term vanish for an arbitrary transformation of coordinate system. The forms Δ^a_b depend on the choice of a coordinate system x^i. The hypothesis 1 is thus falsified (unless we want to fix not only "our" frame E_a, but also "our" coordinate system. If we do this, however, we move from general relativity

to special relativity. We do not bother anymore about general coordinate covariance. That IS a possibility, but I am not sure if this is the path we want to follow. I do not think that this is what Shipov intended. Well, in fact, even this is not a viable possibility. Why? Because to get rid of unwanted term in Eq. (25) we must restrict ourselves only to linear transformations (as then Jacobi matrix is a constant, and its derivatives vanish). But we do not have a constant metric, so we are not in special relativity situation.
Summarizing: The forms $\Delta^a{}_b$ depends on a coordinate system x^i, therefore they can not be considered as forms defining a connection with respect to the vierbein E_a.

But, wait! If the forms $\Delta^a{}_b$ change with the change of coordinate system, then perhaps hypothesis 2 can be proven to hold. In fact, Eq. (25) looks very much similar to Eq. (11). So, maybe this is the right interpretation? We will see in the next section, that also this interpretation fails. It fails by a sign!

4 Vector fields, one-forms,connections and coordinates

Before proceeding with our calculations it is necessary to comment upon multifaceted nature of geometrical objects we are dealing with, and upon coordinate and coordinate-free ways of working with these objects.

In differential geometry we are studying geometrical objects that live on differentiable manifolds. What is a differentiable manifold? It is an abstract mathematical object. It is defined as a set of points M, endowed with certain structure: with a differentiable structure. What is a differentiable structure? There are several, equivalent, definitions. Usually a differentiable structure is defined as an "atlas of compatible, local, coordinate systems - whatever it means (see textbooks books on differential geometry). Even if a differentiable structure itself is defined through coordinate systems, there is a general tendency in differential geometry to deal with coordinate-free description, as long and as much as possible. The rule of the thumb is, however, that all good geometers check there theorems, work out there examples, get new induitions about relationships, doing coordinate calculations (and they all know how to calculate using coordinates), and THEN express the result using coordinate-free language.

Differential geometry deals with geometrical objects and "natural operations" on geometrical objects. What is a "natural operation?" After getting some experience, after going through the proofs of many theorems, after studying examples, one gets a pretty good feeling about it. Then one can go even higher and learn the definition of a "natural operation" - but this is difficult and not in standard textbooks.

What are geometrical objects? Manifold itself is a geometrical object. Differentiable function on a manifold is a geometrical object. Differentiable map from one manifold to another is a geometrical object. Point P of a manifold M is a geometrical object. But most of differential geometry deals with vector fields, differential forms, tensors, connections. Usually we start with definition of a vector. Or, more precisely, with definition of "vector v tangent to M at P". There are several definitions of this concept - all of them are equivalent. To prove that they are equivalent needs work, and usually students of a good course of differential geometry are forced to do these proofs themselves - which is a VERY useful exercise). The standard, "old fashioned", definition is that a vector is a set of n numbers in each coordinate system, with a linear transformation law under coordinate transformations. This is how geometrical objects are defined in J. A. Schouten and many other older textbooks. Another standard definition (see for instance S. Sternberg)is that tangent vector at P is "an equivalence class of curves through P that are tangent to each other (of course one defines first what is a differentiable curve and what it means that two curves are tangent to each other). Yet another definition (R.L. Bishop and S.I. Goldberg) is that tangent vector at P is a "differentiation of the algebra of funcions at P". The first definition makes an explicit use of coordinates, while in the two other definitions use of coordinates is hidden. For instance, in the second definition, one first uses coordinates to define a diffentiable curve, and what it means that two curves are tangent at a point, but, once this is done,then tangent vector is defined in such "simple and elegant way!" Look: "it is an equivalent class if differentiable curves tangent at P." Am I using coordinates? No! It is very important to be able able to show that all these definitions are equivalent. Only then one gets a feeling of what differential geometry is about.

Once we have vectors, we can consider operations on vectors. The simplest operation is that of addition. How we define addition of vectors? If we use our "old fashioned" definition, then we do as follows (I will do it step by step)
Let u and v be tangent vectors at P.

Step 1: Choose a coordinate system x^i around P - then u and v are represented, with respect to this coordinate system, by their n components: u^i and v^i respectively. Then define, in this coordinate system x^i n numbers w^i by $w^i \equiv u^i + v^i$.
Step 2. Now prove the following
Lemma: If w^i are defined through the above algorithm in every coordinate system, then they transform like a vector.
Proof. Let $x^{i\prime}$ be another coordinate system. We want to prove vector law, that is we want to prove that

$$w^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} w^i. \tag{26}$$

Let us compute. According to our prescription we have $w^{i\prime} = u^{i\prime} + v^{i\prime}$. Because u and v are vectors, by our assumption, we have:

$$u^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} u^i. \tag{27}$$

and

$$v^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} v^i. \tag{28}$$

Substituting:

$$w^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} u^i + \frac{\partial x^{i\prime}}{\partial x^i} v^i \tag{29}$$

Now, notice that that we can factor the term $\frac{\partial x^{i\prime}}{\partial x^i}$ out, because it is the same in both terms:

$$w^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} (u^i + v^i) \tag{30}$$

Now we notice that $(u^i + v^i)$ is nothing but our w^i, thus

$$w^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} w^i \tag{31}$$

QED We are done. We have seen that by adding components of two vectors, doing it in every coordinate system, we produce a new vector. We call this new vector the sum of vectors u and v, and denote it by $u + v$.

That was easy. Now, suppose, we are brave, and we want to define also "product of two vectors". Our prescription follows the above: In each coordinate system define numbers w^i as $w^i = u^i \cdot v^i$, where $\cdot$ stands for multiplication. Can we prove now that we have defined a new vector w? Can we prove, again, that

$$w^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i} w^i \tag{32}$$

No. The trick with factoring out the Jacobian will not work this time! We will get

$$w^{i\prime} = \frac{\partial x^{i\prime}}{\partial x^i}\frac{\partial x^{i\prime}}{\partial x^j} u^i v^j \tag{33}$$

and that is how far we can go. $w^{i\prime}$ can not even be expressed as a function of w^i!.

The above simple example explains how we check whether a certain operation produces, out of available geometrical objects, a new geometrical object. Some operations do that. Some other don't

What geometrical operations on vectors can we perform? We can add them, substract, multiply by scalars. We can also take tensor product of vectors to produce contravariant tensors. Thus, for instance, if u and v are vectors, then $u \otimes v$ is a contravariant tensor of second order. In coordinates:

$$(u \otimes v)^{ij} = u^i v^j \tag{34}$$

These are operations on vectors at a point. When we have *vector fields*, then we have one more operation, that of a Lie bracket. It is rather difficult to describe Lie bracket in a coordinate free way. Therefore it is usually defined by the following theorem:

Theorem Let u and v be two vector fields. In each coordinate system x^i define numbers w^i by the formula

$$w^i = u^k \frac{\partial v^i}{\partial x^k} - v^k \frac{\partial u^i}{\partial x^k} \tag{35}$$

Then the numbers w^i transform by the vector transformation law (33) when going from one coordinate system to another.

Proof Goes by explicit calculation as in our example of vector sum. But this time involves cancellation of second order derivatives from the two terms of Eq. (35)

Once this theorem is proved (and not before!) the definition comes: As the theorem shows, w^i defined above are components of a vector field w. We call this vector field Lie bracket (or commutator) of u and v and denote it by $[u, v]$. Thus, by definition:

$$([u, v])^i = u^k \frac{\partial v^i}{\partial x^k} - v^k \frac{\partial u^i}{\partial x^k}. \tag{36}$$

Notice that we could try to define, in every coordinate system,

$$w^i = u^k \frac{\partial v^i}{\partial x^k} \tag{37}$$

and this would be a perfectly good definition, but so defined numbers w^i would not define a vector field, they would not transform into themselves, even in some ugly way! Like it was when we tried to define product of two vectors (but tensor product is okay).

Let us now consider differential forms. 0-forms are just functions. One-forms are "covariant vectors", thus if ω is a one-form, its coordinates with respect to a coordinate system x^i are denoted ω_i, and with a coordinate change we must have

$$\omega_{i\prime} = \frac{\partial x^i\prime}{\partial x^{i\prime}} \omega_i \tag{38}$$

Only when the above law holds, we know that the numbers ω_i and $\omega_{i\prime}$, given in coordinate system x^i and $x^{i\prime}$ respectively, describe *the same geometrical object: one form* ω.

Forms can also be defined in coordinate-free language, Once we have done our work of defining vectors, vector-fields, and contravariant tensors, we can use formalism developed in linear algebra to deal with forms. Linear algebra has the concept of "dual vector space" to a given space, and thus differential 1-forms can be defined as "vectors in the dual space" - they are linear functionals on vectors (or on vector fields). If ω is a 1-form, and if v is a vector, then $\omega(v)$ is a scalar - it is the value of the linear functional ω on the vector v. If x^i is a coordinate system, then let ∂_i denote vector fields tangent to the coordinate lines. Evaluating our ω on these vector fields we get n scalars $\omega_i = \omega(\partial_i)$. This formula connects coordinate-free, and "covariant tensor of rank 1 definitions of a 1-form. For an arbitrary vector v we have

$$\omega(v) = \omega_i v^i, \tag{39}$$

where summation over the repeated index is performed. If we use coordinate do not have to prove any more that we can add two differential forms, and that we get again a differential form. This is proven in linear algebra for general dual vector spaces. What operations on differential forms can we perform? We can perform tensor product, as for vectors, but in the course of history it was realized that if we want to be able to "differentiate" covariant tensors, when no extra structure on the manifold is given, then such an operation could not be defined on arbitrary covariant tensors! Only for antisymmetric covariant tensors such an operations could be found. Therefore a differential form of order p is defined as antisymmetric covariant tensor of rank p. Of course, as always, there are several other, equivalent, definitions. For instance we could first consider antisymmetrized tensor products of vectors (Grassmann algebra), and then define differential forms as dual objects. Result would be the same. What geometrical operations on differential forms can be performed on differential forms? Given two differential forms we can add them, subtract, multiply by scalars, we can also take wedge product. All these operators can be expressed in coordinates. Sometimes they are defined through coordinate formulas (and then we always need a proof that that the result of this operation is a differential form - that is that it transforms like a differential form, under transformation of coordinates). Sometimes they are defined using techniques developed in multilinear algebra, so theorems from algebra are used to define these operations. We also have exterior derivative d, and this is defined first in coordinates, and then proven to give again a differential form (but of a higher rank). Finally there is an operation of inserting a vector into a differential form, which decreases the rank by one. We can define it using ccordinate language (but then we have to prove that the result transforms as a differential form), or we can use multilinear algebra formalism and write:

$$i(v)\omega(w_1, \ldots, w_{p-1}) \equiv \omega(v, w_1, \ldots, w_{p-1}). \tag{40}$$

From d and i we can build other operators, like, $i(v)d$, $\mathrm{d}i(v)$, and Lie derivative operator $L_v = i(v)d + di(v)$.

When we are given some vectors, and some differential forms, how do we build other vectors and other differential forms from them? There are two ways: either we use geometrical operators that are known to lead from geometrical objects to geometrical objects (in our case: sums, tensor products, Lie brackets, wedge products, exterior derivatives, insertions), and then we always know that what we get is another geometrical object, or - we can decide to go for an adventure and use coordinates. In this case we always have to check that what we get is a geometrical object of the kind we want. Sometimes we do this because we have some intuition telling us that it is so, but we can not easily formulate our prescription in terms of coordinate-free operations that we know. Sometimes later on we find that our formula can be also expressed in coordinate free way, and therefore our proof of the correct transformation law is no longer necessary. Sometimes we hope, in this way, to discover a new object that can not be expressed in terms of elementary coordinate-free operations! But then the burden of proving that what we have is a geometrical object of the kind we claim - lies on us!

Equipped with all this knowledge, let us consider now the situation give to us with Shipov Eq. (5.67). What are the objects that we have to our disposal? We are given n 1-forms e^a, and we are given n vectors (or, rather, vector fields) E_a. They are in duality relation: $e^a(E_b) = \delta^a_b$, or, using insert notation $i(E_b)e^a = \delta^a_b$. We do not need coordinates to express these duality relations. What we want to build from these building blocks? We want to build n^2 differential 1-forms. How can we do it? As discussed above, there are two ways: either we exploit all available coordinate-free geometric operators, and see what we can do with them, or we take the adventure path: we use coordinates, and we take on ourselves the burden of a proof that what we obtain are differential forms, that is that they transform in a appropriate way under coordinate transformations - as it was when we had to prove that sum of two vectors gives again a vector!

Let us consider first the coordinate-free method. We are given E_a and we are given e^b and we want to construct differential forms $\Delta^a{}_b$. What would be the simplest solution to our problem? It often happens that *the simplest* solution escapes our attention. In this case the simplest solution is: Set $\Delta^a{}_b$ all to be zero! Indeed, this statement is independent of any coordinate system. We learn in algebra that there is such a thing as "zero vector", and that being a zero vector is independent of any coordinate system. Is our solution trivial? By no means. $\Delta^a{}_b = 0$ describe affine connection of absolute parallelism - the affine connection discussed by Cartan, by Einstein, by Shipov before. It is the connection in which the frame E^a is *parallel*. Why we din not see this before? Because Shipov was writing coefficients of this connection not with respect to E_a but with respect to the coordinate frame ∂_i - he called these coefficients $\Omega^i{}_j$. It is seen from the formula (5) that if connection coefficients are zero in one frame - they are non-zero in another frame!

So, we have the simplest solution to our problem - and we even understand what it is - it is the connection of distant parallelism. Zero curvature, non-zero torsion. We got warmed up and we are willing to find another solution.

This time a non-zero solution. If we find one - it will be a different connection. Possibly it will be THE SHIPOV CONNECTION! So, let us try. What about the formula:

$$\Delta^a_{\ b} = i(E_b)e^a \tag{41}$$

No, this we know are scalars δ^a_b, while we need 1-forms! But what about

$$\Delta^a_{\ b} = i(E_b)de^a \tag{42}$$

This sounds good! We have n^2 differential 1-forms, and they are, in general, non-zero. Also, they have been built using geometrical operators, so we do need to check that they define forms. We DO HAVE A NEW CONNECTION! But is it really "new"? And how it relates to Shipov's formula (5.67). Before discussing these questions, let us notice that, in fact, we have not one, but whole one-parameter family of connections, as we can set

$$\Delta^a_{\ b}(\lambda) = \lambda i(E_b)de^a, \tag{43}$$

where λ is a real number $0 \leq \lambda \leq 2$. (We could take λ being an arbitrary real number, not necessarily restricted by being from the interval $[0, 2]$, but it is sufficient to take it from this interval, as all interesting things happen just there.... .) For $\lambda = 0$ we get our old friend, the zero connection, but for λ different from zero we have something new. Or, is it really new? To understand what we are doing it is useful to apply some standard results from books on differential geometry. They are simple to derive, so usually they are left as exercises. These results are rather difficult to derive using differential forms, therefore they are usually in books stated in terms of connection being defined as "covariant derivative ∇ of vector fields". If X and Y are two vector fields, then $\nabla_X Y$ denotes the third vector field, constructed out of X and Y - the covariant derivative of Y with respect to X. This is also an intrinsique, geometrical coordinate free language. I will use the textbook by Bishop and Goldberg to formulate these results.
First of all, in Ch. 5.10, Eqs. $(5.10.2), (5.10.3)$ torsion S of an arbitrary connection ω is defined (with respect to a moving frame E) as

$$S = 2(de^a + \omega^a_{\ b} \wedge e^b) \otimes E_a \tag{44}$$

, where I adjusted the notation to our conventions. Torsion is a vector-valued two-form. What is the torsion form of our "zero connection?" By putting $\omega_a^{\ b} \equiv 0$ in the above formula we get

$$S = 2de^a \otimes E_a \tag{45}$$

which is similar, but yet different to our Eq. (42). In torsion formula we have tensor product (torsion is vestor valued 2-form, while in Eq. (42) we have scalar valued 2-forms! Yet there is some similarity!
Then, we have
Problem 5.10.1 If ∇ is a connection on M then the *conjugate connection* $D^\star$ is defined by:

$$\nabla_X Y = \nabla_X Y + T(X,Y), \tag{46}$$

where T is the torsion of ∇. Show that $\nabla^\star$ is actually a connection on M and that the torsion of $\nabla^\star$ is $-T$. Hmm.... this seems to be interesting.... Is this "conjugate connection" the same as Kiehn calls "left connection?", and how it relates to what we have? We have already calculated the torsion S of the zero connection. This is given by Eq. (45). Let us now calculate torsion of the connection $\Delta^a{}_b(2)$, that is, by the connection $\Delta^a{}_b(\lambda)$ - at the other end of the interval $[0,2]$. Let us use formula (44), replacing $\omega^a{}_b$ by $i(E_b)de^a$. Then we have

$$S(\lambda=2) = 2(de^a + 2(i(E_b)de^a)\wedge e^b)\otimes E_a. \tag{47}$$

We need to compute the second term, which seems to be a little bit complicated, and perhaps can be simplified? Here we need a little lemma. We know that, in coordinate notation, the identity $E^i{}_b e^b{}_j = \delta^i_j$ holds. Can we express this identity in terms of a coordinate free operation on differential forms? The answer is "yes". We have:

$$e^b i(E_b) = Id \tag{48}$$

-the identity operator! Knowing this, let us consider the second term in the bracket in Eq. (47). Wa have:

$$2(i(E_b)de^a)\wedge e^b = -2\wedge e^b(i(E_b)de^a) = -2Idde^a = -2de^a, \tag{49}$$

where the minus sign follows from the fact that we are changing the order of two one-forms entering the wedge product.Substituting into Eq. (47) we get

$$S(\lambda=2) = 2(de^a + 2(i(E_b)de^a)\wedge e^b)\otimes E_a = 2(de^a - 2de^a)\otimes E_a \tag{50}$$

so that

$$S(\lambda=2) = -2de^a\otimes E_a = -S(\lambda=0). \tag{51}$$

Surprise! The connection $\Delta^a{}_b(\lambda=2) = 2de^a\otimes E_a$ has torsion which is equal minus torsion of the zero connection! Is this connection really the "conjugate" conjugation introduced in Bishop and Goldberg Problem 5.10.1? I will

skip the proof that it is really so. But what about our first guess, connection $\Delta^a{}_b(\lambda = 1)$, defined in Eq. (42)? This connection is in the middle, between the two extremes.... Well, we already know what will happen. This connection will have zero torsion!. In fact, we find, again in Bishop and Goldberg, the following problem:
Problem 5.10.3 If ∇ is a connection on M, show that ${}^sD = \frac{1}{2}(\nabla + \nabla^\star)$ is symmetric and find its coefficients with respect to a coordinate basis in terms of the coefficients of ∇. The connection ${}^s\nabla$ is called the symmetrization of ∇.

Here "symmetric" means torsion zero! So, maybe THIS IS WHAT SHIPOV means in his equation 4.57? Shipov is using coordinates, we were "good", we were not using coordinates. We were operating on differential forms! But to compare our formula with Shipov's formula, we have to write our formula (42) in coordinates too. So, let us do it:

$$\Delta^a{}_b(\lambda)_i = \lambda E^j{}_b(de^a)_{ji} = \lambda E^j{}_b(\partial_j e^a{}_i - \partial_i e^a{}_j) \tag{52}$$

and this is all that we get. To compare with Shipov's 4.57, or our Eq. (8), we use Leibnitz rule to move differentiation to E_b:

$$\Delta^a{}_b(\lambda)_i = \lambda(-(\partial_j E^j{}_b)e^a{}_i + (\partial_i E^j{}_b)e^a{}_j) \tag{53}$$

or, returning to differential form notation (that is skipping the index i:

$$\Delta^a{}_b(\lambda) = \lambda(-(\partial_j E^j{}_b)e^a + (dE^j{}_b)e^a{}_j) \tag{54}$$

What we see is this: for $\lambda = 1$ the second term is exactly our Eq. (8), but there is also the first term! Can we skip the first term? The answer is "no". The term is there for a reason. The same way we can not skip the first term in the formula for Lie bracket of vector fields Eq. (35): this term is necessary, without this term the result is not a vector field! The same with Eq. (54) - without the first term the result is not a form! Thus Shipov's formula (5.47) does not define a connection! There is a term lacking in his formula. Then the following question arises: has this term been omitted by mistake or by purpose? And if by purpose, the for what purpose? To mislead the reader that have a blind faith in his equations? Thsi we do not know. But we are not finished yet. We have discussed torsion in some length. What about curvarture?