Quantum fractals on n–spheres

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Abstract

Using the Clifford algebra formalism we extend the quantum jumps algorithm of the Event Enhanced Quantum Theory (EEQT) to convex state figures other than those stemming from convex hulls of complex projective spaces that form the basis for the standard quantum theory. We study quantum jumps on n-dimensional spheres, jumps that are induced by symmetric configurations of non-commuting state monitoring detectors. The detectors cause quantum jumps via geometrically induced conformal maps (Möbius transformations) and realize iterated function systems (IFS) with fractal attractors located on n-dimensional spheres. We also extend the formalism to mixed states, represented by "density matrices" in the standard formalism, (the n-balls), but such an extension does not lead to new results, as there is a natural mechanism of purification of states. As a numerical illustration we study quantum fractals on the circle (one-dimensional sphere and pentagon), two–sphere (octahedron), and on three-dimensional sphere (hypercube-tesseract, 24 cell, 600 cell, and 120 cell). The attractor, and the invariant measure on the attractor, are approximated by the powers of the Markov operator. In the appendices we discuss the Hamilton's "icossian calculus" and its application to quaternionic realization of the binary icosahedral group that is at the basis of the 600 cell and its dual, the 120 cell.

1 Introduction

> "The accepted outlook of quantum mechanics (q.m.) is based entirely on its theory of measurement. Quantitative results of observations are regarded as the only accessible reality, our only aim is to predicts them as well as possible from other observations already made on the same physical system. This pattern is patently taken over from the positional astronomer, after whose grand analytical tool (analytical mechanics) q.m. itself has been modelled. But the laboratory experiment hardly ever follows the astronomical pattern. The astronomer can do nothing but observe his objects, while the physicist can interfere with his in many ways, and does so elaborately. In astronomy the time–order of *states* is not only of paramount practical interest (e.g. for navigation), but it was and is the only method of discovering the *law* (technically speaking: a hamiltonian); this he rarely, if ever, attempts by following a single system in the time succession of its states, which in themselves are of no interest. The accepted foundation of q.m. claims to be intimately linked with experimental science. But actually it is based on a scheme of measurement which, because it is entirely antiquated, is hardly fit to describe any relevant experiment that is actually carried out, but a host of such as are for ever confined to the imagination of their inventors."

So wrote Ervin Schrödinger fifty years ago [1]. Today the standard scheme of q.m. is as antiquated as it ever was, and provides no answer to the most fundamental questions such as "what is time?", and how to describe *events* that happen in a single physical system, such as our Universe.[1] The present paper follows the line of ideas developed in a series of papers that has led to the Event Enhanced Quantum Theory (EEQT), as summarized in [3], and recently extended in [4], but we now go beyond that framework. While, following von Neumann, we keep the algebraic structure as one of the most important for the mathematical formalism of q.m., and we propose to dispose of the concept of "observables" and of "expectation values" at the

[1] Nowadays the defenders of the "antiquated scheme" of q.m. go as far as to assign "crackpot index" to those who question this scheme. So, for instance, 10 points (on the scale of 1–50), are assigned for each claim that quantum mechanics is fundamentally misguided, and another 10 points for arguing that while a current well-established theory predicts phenomena correctly, it doesn't explain "why" they occur, or fails to provide a "mechanism" [2].

fundamental level. We also dispose of the concept of "time", understood as a "continuous parameter", external to the theory. Our philosophy, concerning "time" is that of the German social philosopher Ernest Bloch:

> "*Zeit ist nur dadurch, daß etwas geschieht und nur dort wo etwas geschiecht.*

So, time is only *then*, when something happens, and only *there* where something happens. Therefore the primary concept is that of an *event*, and of the *process* - that is a sequence of events. Time, as a continuous, global variable, comes in only in the limit of a large number of events. The primary process is that of "quantum jumps". It is an irreversible process in an open system, and every system in which the "future" is only "probable", rather than determined, is necessarily an open system. The mathematical formalism of the standard quantum theory is based on complex Hilbert spaces and Jordan algebras of self–adjoint operators. It involves interpretational axioms for expectation values and eigenvalues of self–adjoint operators as "possible results of measurements", yet it does not provide a framework for *defining the measurements* [5, 6]. In view of these considerations, Gell-Mann would certainly score a high crackpot index [2] for this statement:

> "Those of us working to construct the modern interpretation of quantum mechanics aim to bring to an end the era in which Niels Bohr's remark applies: 'If someone says that he can think about quantum physics without becoming dizzy, that shows only that he has not understood anything whatever about it'."

The same can be said about the last paragraph of Schrödingers paper [1], where he wrote

> "We are also supposed to admit that the extent of what is, or might be, observed coincides exactly with what quantum mechanics is pleased to call observable. I have endeavored to adumbrate that it does not. And my point is that this is not an irrelevant issue of philosophical taste; it will compel us to recast the conceptual scheme of quantum mechanics."

The need for an open–minded approach is well noted by John A. Wheeler, who ends his book "Geons, Black Holes & Quantum Foam" [7] with the following quote from Niels Bohr's friend Piet Hein:

I'd like to know

what this whole show
is all about
before it's out.

Alain Connes and Carlo Rovelli [8] proposed to explain the classical time parameter as arising from the modular automorphism group of a KMS state on a von Neumann algebra over the field of complex numbers $\mathbb{C}$.[2] But their philosophy applies, at most, to equilibrium states, while "quantum foams" before the Planck era are certainly far from equilibrium. David Hestenes [10, 11] proposed to understand the role of the complex numbers in quantum theory in terms of the Clifford algebra. This is also our view. L. Nottale, in his theory of "scale relativity" [12] proposed an alternative idea, where the complex structure arises from a stochastic differential equation in a fractal space–time. We think that our approach may serve as a connecting bridge between fractality, the nontrivial topology of dodecahedral models of space–time, as discussed by J–P. Luminet et al. [13] (cf. also [14].), and the late thoughts of A. Einstein [15, p. 92], who wrote:

> "To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraical method of description of nature, that is to the elimination of continuous functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At the present time, however, such a program looks like an attempt to breathe in empty space."

The present paper is a technical one. It fills the empty space with discrete structures, and it deals with the discrete random aspects of quantum jumps generated by the algebraic structure of real Clifford algebras of Euclidean spaces, and of their conformal extensions. The jumps are generated by Möbius transformations and lead to iterated function systems with place dependent probabilities, thus to fractal patterns on n–spheres. Our ideas are close to those of W. E. Baylis, who also noticed [16] the similarities

[2] C.f. also [9], where a similar idea, based on a KMS equilibrium state is discussed in a broader, philosophical framework

between the Clifford algebra scheme and the formal algebraic structure of q.m.

In Sec. 2 we introduce our notation, which follows the one of Deheuvels [17]. In Proposition 1 we recall the vector space isomorphism between the Clifford algebra and the exterior algebra, and in Proposition 2 we define the trace functional, and list its properties that are important for applications to quantum probabilities.

In Sec. 3 we discus Möbius transformations of the spheres S^n, as well as their natural extensions to their interiors B^{n+1}. Proposition 3 gives the explicit form of the embedding of the Clifford algebra $C(1,n+1)$ in the matrix algebra $Mat(2, C(n+1))$ and allows us to realize the group $Spin(1, n+1)$ by two–by–two diagonal matrices with entries in $\mathcal{C} = C(n+1)$. Proposition 4 provides the main result of this section: to every non–zero vector in the interior of the unit ball $B^{n+1} \subset E = E^{n+1}$, written as $\epsilon\mathbf{n}$, $0 < \epsilon < 1$, $\mathbf{n}^2 = 1$, we associate (cf. Eq. (3.12) the element $g(\epsilon\mathbf{n}) \in Spin(1, n+1)$, that defines a Möbius transformation $g(\epsilon\mathbf{n})$ of S^n and its extension to $\bar{B}^{n+1}$. We give the explicit form of these transformations (cf. Eq. (3.15) and calculate the Radon-Nikodym derivatives of the transformed surface and volume areas (cf. Eqs. (3.18), (3.19)).

In Sec. 4 we discuss iterated function systems (IFS) of conformal maps and introduce the important concept of the Markov operator which is later being used in our numerical simulations (cf. Sec. 5). Proposition 5 of this section is important in applications to quantum theory. One of the most important features of the standard, linear, quantum mechanics is the fact that "observables" are restricted to bilinear functions on pure states. Therefore different mixtures of pure states leading to the same "density matrix" are claimed to be experimentally indistinguishable. In our Proposition 5, and in Corollary 1, we show that if the probabilities of the iterated function systems of Möbius transformations are given by geometrical factors derived from the maps themselves (cf. Eqs. (3.15),(4.31)), and also the additional balancing condition (4.30)), then the Markov operator restricts to the space of functions on S^n given by the trace on the Clifford algebra, thus leading to a linear Markov semi-group. Corollary 2 gives the explicit form of the Markov operator for the case when the IFS of Möbius transformations is endowed with geometrical probabilities given by Eq. (4.31).

Sec. 5 contains the results of the numerical simulations of IFS of Möbius transformations that lead to "quantum fractals". We study quantum fractals on the circle (one-dimensional sphere and pentagon), two–sphere (octahedron), and on three-dimensional sphere (hypercube-tesseract, 24 cell, 600 cell, and 120 cell). The last section contains the summary and conclusions

and also points out some open problems.
In the appendices we discuss the Hamilton's "icossian calculus" (in particular we quote *in extenso* the original Hamilton's paper published in 1856), and its application to quaternionic realization of the binary icosahedral group that is at the basis of 600 cell and its dual, the 120 cell.

2 Notation

We denote by $E^{(r,s)}$ the real vector space $\mathbb{R}^n$, $n = r+s$, endowed with the quadratic form $q(x)$ of signature (r,s). $E^n = E^{(n,0)}$ is the standard $n-$dimensional Euclidean space. The Clifford algebra of $E^{(r,s)}$ is denoted by $C(E^{(r,s)})$ or, shortly, as $C(r,s)$. The Clifford map $E^{(r,s)} \ni x \mapsto \phi(x) \in C(E^{(r,s)})$ satisfies $\phi(x)^2 = q(x)I$. x and $\phi(x)$ are often identified, so that $E^{(r,s)}$ can be considered as a vector subspace of $C(r,s)$ that generates $C(r,s)$ as an algebra. The principal automorphism of $C(E^{(r,s)})$ is denoted by π and is determined by $\pi(x) = -x$, $x \in E^{(r,s)}$, while the principal anti–automorphism τ, denoted also as $\tau(a) = a^\tau$, is determined by $x^\tau = x$, $x \in E^{(r,s)}$. Their composition ν is the unique anti–automorphism satisfying $\nu(x) = -x$ for all $x \in E^{(r,s)}$. $\mathbb{C}(n)$ (resp. $\mathbb{R}(n)$) will denote the algebra of complex (resp. real) matrices $n \times n$.
Let us recall that, as a vector space, Clifford algebra is naturally graded and isomorphic the exterior algebra. In particular we have the following result :

Proposition 1. *Let e_i, $i = 1, 2, \dots, n$ be an orthonormal basis for $E^{(r,s)}$, and let $e_I : I = (i_1, i_2, \dots, i_p)$, $1 \le i_1 < i_2 < \dots < i_p \le n$ be defined as the Clifford products $e_I = e_{i_1} e_{i_1} \dots e_{i_p}$, with $e_I = 1$ for $I = \emptyset$. Then the set $\{e_I\}$ of 2^n vectors in $C(r,s)$ is a linear basis of $C(r,s)$, the subspaces C_p generated by e_I, $I = (i_1, \dots i_p)$ are independent of the choice of the orthonormal basis e_i, and $C(r,s)$ is the direct sum of vector subspaces C_p :*

$$C(r,s) = \bigoplus_{k=0}^{n} C_p \tag{2.1}$$

Moreover, for each $p = 0, \dots, n$ the skew–symmetric map α_p from $E^{(r,s)} \times E^{(r,s)} \times \dots \times E^{(r,s)}$ (p times) to $C(r,s)$ given by:

$$\alpha_p(x_1, x_2, \dots, x_p) = \frac{1}{p!} \sum_\sigma (-1)^\sigma x_{\sigma 1} x_{\sigma 2} \dots x_{\sigma p},$$

determines an isomorphism of the vector subspace $\Lambda^p E(r,s)$ of the exterior algebra $\Lambda E(r,s)$ onto C_p that sends $e_{i_1} \wedge \dots \wedge e_{i_p} \in \Lambda^p E(r,s)$ to $e_{i_1} \dots e_{i_p} \in C_p \subset C(r,s)$.

Proof: c.f. [17, Theoreme VIII.10] ■

We denote by Φ the linear functional on $C(r,s)$ assigning to each element $a \in C(r,s)$ its scalar part $\Phi(a) = a_0 \in C_0$ in the decomposition (2.1). Then the following proposition holds:

Proposition 2. *The functional Φ has the following properties:*

(i) $\Phi(1) = 1$,

(ii) $\Phi(a^\tau) = \Phi(a), \ \forall a \in C(r,s)$,

(iii) $\Phi(ab) = \Phi(ba), \ \forall a, b \in C(r,s)$,

(iv) $(a,b) \doteq \Phi(a^\tau b)$ *is a nondegenerate, symmetric, bilinear form on $C(r,s)$, that is positive definite if the original quadratic form on E is positive definite. We have $\Phi(a) = (1,a) = (a,1)$, $\forall a \in C(r,s)$.*

(v) $(ab,c) = (b, a^\tau c) = (a, cb^\tau), \ \forall a,b,c \in C(r,s)$.

Proof: (i) and (ii) follow immediately from the definition. In order to prove (iii) notice that if $\{e_i\}$, $i = 1, \ldots, n$ is an orthonormal basis in $E^{(r,s)}$, $\{e_I\}, I = \{i_1 < \ldots < i_p\}$ is the corresponding basis in $C(r,s)$, and $a = \sum_I a_I e^I$, $b = \sum_I b_I e_I$ are the decompositions of a and b in the basis e_I, then $\Phi(ab) = \sum_I a_I b_I \Phi(e_I e_I) = \Phi(ba)$. From the very definition of the scalar product (a,b) it follows that $(a,b) = \Phi(a^\tau b) = \Phi((a^\tau b)^\tau) = \Phi(b^\tau a) = (b,a)$. Moreover, we have (e_I, e_J)=0 if $I \neq J$, and also $(e_I, e_I) = {e_{i_1}}^2 \ldots {e_{i_p}}^2 = (-1)^{s(I)}$, where $s(I)$ is the number of negative norm square vectors in I. In particular e_I is orthonormal with respect to the scalar product in $C(r,s)$, and so (iv) holds. We have $(ab,c) = \Phi((ab)^\tau c) = \Phi(b^\tau a^\tau c) = (b, a^\tau c) = \Phi(a^\tau c b^\tau) = (a, cb^\tau)$, which establishes (v). ■ [3]

3 Möbius transformations of S^n and their extensions to B^{n+1}

The unit n–sphere S^n, that is the boundary of the closed unit ball $\bar{B}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$, can be considered as a one-point compactification of $\mathbb{R}^n$. S^n is also the Möbius space of $\mathbb{R}^n$ where Möbius transformations

[3] It is easy to see that $\Phi(a) = (1/2^n)\mathrm{tr}(l(a))$, where $l(a)$ is the left multiplication by a acting on $C(r,s):\ l(a)b = ab$, and the trace is taken over $C(r,s)$, see e.g. [18, p. 601] for a general discussion. Because of this property Φ may be called a *trace functional.*

are realized as pseudo–orthogonal transformations of $\mathbb{R}^{(1,n+1)}$ - cf. e.g. [19, Theorem 2.2.1.3.3]. We will discuss a special class of these transformations and their natural extension to the interior of $\bar{B}^{n+1}$. To simplify the notation, in what follows, we will denote $\mathbb{R}^{n+1}$ by E, and we will set its vectors in bold, as, for example $\mathbf{x}, \mathbf{r}, \mathbf{n}$, etc. The natural quadratic form and the associated bilinear form in E will be denoted as $\mathbf{x} \mapsto \mathbf{x}^2$ and $(\mathbf{x}, \mathbf{x}') \mapsto \mathbf{x} \cdot \mathbf{x}'$ respectively.

Let B^{n+1} be the open unit ball in E, let $\bar{B}^{n+1}$ be its closure, and S^n its boundary:

$$\bar{B}^{n+1} = \{\mathbf{x} \in E : \mathbf{x}^2 \leq 1\}, \quad B^{n+1} = \{\mathbf{x} \in E : \mathbf{x}^2 < 1\}, \tag{3.2}$$

and

$$S^n = \{\mathbf{x} \in E : \mathbf{x}^2 = 1\}. \tag{3.3}$$

Let

$$E^{(1,n+1)} \doteq \mathbb{R} \oplus E$$

be equipped with the quadratic form

$$q(x^0 \oplus x) = q(x^0, \mathbf{x}) = (x^0)^2 - \mathbf{x}^2, \qquad x_0 \in \mathbb{R}, \mathbf{x} \in E,$$

where, following the standard notation for the Minkowski space, we denote by x^0 the additional $(n+1)$'s coordinate. Let

$$C^+ = \{(x^0, \mathbf{x}) \in E^{(1,n+1)} : x^0 > 0, \, \mathbf{x}^2 - (x^0)^2 = 0\}$$

be the positive isotropic cone of $E^{(1,n+1)}$, and denote by T_1 the hyper–plane $T_1 = \{(\mathbf{x}, x^0) : x^0 = 1, \, \mathbf{x} \in E\}$. Then the intersection of T_1 with C^+ can be identified with S^n, and the intersection of T_1 with the interior region of C^+ can be identified with B^{n+1} :

$$\bar{B}^n = \{(x^0, \mathbf{x}) : \mathbf{x}^0 = 1, \, q(x^0, \mathbf{x}) \geq 0\},$$

$$B^{n+1} = \{(x^0, \mathbf{x}) : \mathbf{x}^0 = 1, \, q(x^0, \mathbf{x}) > 0\},$$

$$S^n = \{(x^0, \mathbf{x}) : x^0 = 1, \, q(x^0, \mathbf{x}) = 0\}.$$

Let $O_{++}(1, n+1)$ be the connected component of the identity of the pseudo–orthogonal group of $E^{(1,n+1)}$. The transformations from $O_{++}(1, n+1)$ act linearly on $E^{(1,n+1)}$ and map bijectively both C_+ and its interior onto themselves. Therefore they induce, by projections, bijections of $\bar{B}^{n+1}$ and of S^n. We will describe now a specific class of these transformations, members of this class being parametrized by vectors $\epsilon\mathbf{n} \in B^{n+1}$.

Let $C(1,n{+}1)$ be the Clifford algebra of $E^{(1,n+1)}$, and let $Spin(1,n{+}1)$ be its spin group[4]. Every element $g \in Spin(1,n{+}1)$ is a product of an even number of positive unit vectors (i.e. $u \in E^{(1,n+1)}$ such that $q(u) = +1$) and an even number of negative unit vectors (i.e. $u \in E^{(1,n+1)}$ such that $q(u) = -1$) – cf. [17, Definition IX.4.C]. Let $\mathcal{C} \doteq C(n+1)$ be the Clifford algebra of E^{n+1}. Notice that $C(1,n+1)$ can be realized as a sub–algebra of the algebra of the algebra of 2×2 matrices with values in $\mathcal{C}$, the Clifford map $E \ni x = (x^0, \mathbf{x}) \mapsto \gamma(x) \in \mathrm{Mat}(2, \mathcal{C})$ being given by:

$$\gamma(x^0, \mathbf{x}) = \begin{pmatrix} 0 & x^0 + \mathbf{x} \\ x^0 - \mathbf{x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^0 + \mathbf{x} \\ \pi(x^0 + \mathbf{x}) & 0 \end{pmatrix}. \tag{3.4}$$

In fact we have the following Proposition[5]:

Proposition 3. *The Clifford algebra $C(1,n+1)$ can be realized as the sub–algebra*

$$A(a,b) = \left\{ \begin{pmatrix} a & b \\ \pi(b) & \pi(a) \end{pmatrix} : a, b \in \mathcal{C} \right\} \tag{3.5}$$

of $Mat(2,\mathcal{C})$. The principal involution π and the principal anti–involution τ of $C(1,n+1)$ can be expressed through their corresponding operations in $\mathcal{C}$ as

$$\pi(A(a,b)) = A(a,-b), \tag{3.6}$$

$$\tau(A(a,b)) = A(\nu(a), \tau((b)). \tag{3.7}$$

The even subalgebra $C_+(1,n+1)$ of $C(1,n+1)$ can then be identified with the set of all $A(a,b)$, with $b = 0$, that is with $\mathcal{C}$.

Proof The first part of the statement follows from the Theorem 2.37 in Ref. [20][6] Let C be the matrix

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.8}$$

[4] We are using the notation conventions of Ref. [17]. In particular the group *Spin* is assumed to be connected (some authors denote it $Spin_0$).

[5] The first part of this Proposition follows from [20, Theorem (2.37), p. 21]

[6] It should be noted that in Ref. [20] the authors use the sign convention that is opposite to ours when defining Clifford algebras of quadratic spaces. Therefore a slight adaptation of their result is needed.

then

$$CA(a,b)C^{-1} = A(a,-b), \tag{3.9}$$

therefore the formula (3.6) defines an involutive automorphism of $C(1,n{+}1)$, and it is clear from (3.4) that it reverses the signs of vectors. Therefore it defines the principal involution of $C(1,n{+}1)$. The map τ given by Eq. (3.7) is an anti–involution, and it is evident from its definition and the Clifford map (3.4) that it leaves vectors unchanged. Therefore it defines the principal anti–involution of $C(1,n+1)$. ■

According to the above Proposition the map $\psi : C_{+}(1,n+1) \ni A \mapsto A_{11} \in \mathcal{C}$ is an algebra isomorphism from the even subalgebra $C_{+}(1,n+1)$ to $\mathcal{C}$. It follows that the group $Spin(1,n+1)$ can be realized via 2×2 matrices $A(g), g \in Spin(1,n+1)$, with values in $\mathcal{C}$, of the form

$$A(g) = \begin{pmatrix} \psi(g) & 0 \\ 0 & \pi(\psi(g)) \end{pmatrix}, \tag{3.10}$$

with $\psi(g) \in \mathcal{C}$, the action of $Spin(1,n+1)$ on $E^{(1,n+1)}$, $g : x = (x^0, \mathbf{x}) \mapsto x' = (x'^0, \mathbf{x}')$ being given by

$$\mathcal{C} \ni (x^0 + \mathbf{x}) \longmapsto (x'^0 + \mathbf{x}') = \psi(g)(x^0 + \mathbf{x})\pi(\psi(g)^{-1}).^7$$

Note: Notice that if the even subalgebra $C_{+}(1,n+1)$ of $C(1,n+1)$ is identified with $\mathcal{C}$, then $\psi(g)$ is identified with g

For each $\mathbf{x} \in \bar{B}^{(n+1)}$ let $P(\mathbf{x})$ be the element of $\mathcal{C}$ defined by

$$P(\mathbf{x}) = (1 + \mathbf{x}). \tag{3.11}$$

Then

a) $P(\mathbf{x})^\tau = P(\mathbf{x})$,

b) $\mathbf{x} = P(\mathbf{x}) - 1$, and

c) $P(\mathbf{x})/2$ is an idempotent if an only $\mathbf{x} \in S^n$.

[7] Using the method indicated in [20, Theorem 6.12] one can show that the image of $Spin(1,n+1)$ by the isomorphism ψ consists of all $a \in \mathcal{C}$ with $\Delta(a) = 1$ and such that $a(x^0 + \mathbf{x})a^\tau = x'^0 + \mathbf{x}'$ for $x^0 \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$.

Notice that S^n is the boundary (and the set of extremal points) of the convex ball $\bar{B}^{(n+1)}$. If $\mathbf{x} \in \bar{B}(n+1)$ is a convex combination of $\mathbf{x}_\alpha$, $\mathbf{x} = \sum_\alpha t_\alpha \mathbf{x}_\alpha$, $0 \leq t_\alpha \leq 1$, $\sum t_\alpha = 1$, then $P(\mathbf{x}) = P(\sum_\alpha t_\alpha \mathbf{x}_\alpha) = \sum_\alpha t_\alpha P(\mathbf{x}_\alpha)$, so that the convex structure of $\bar{B}^{(n+1)}$ is being mirrored by the convex structure in the algebra $\mathcal{C}$.

Proposition 4. *For each $0 < \epsilon < 1$ and each $\mathbf{n} \in S^n \subset E$, let $g(\epsilon\mathbf{n})$ be the element of the Clifford algebra $C(1, n+1)$ defined by:*

$$g(\epsilon\mathbf{n}) = \frac{(e_0 + \epsilon\mathbf{n})(e_0 - \epsilon\mathbf{n})}{1 - \epsilon^2}, \tag{3.12}$$

where e_0 is the vector $(1, \mathbf{0}) \in E^{(1,n+1)}$.

Then $g(\epsilon\mathbf{n}) \in Spin(1, n+1)$,

$$g(\epsilon\mathbf{n})^{-1} = \frac{(e_0 - \epsilon\mathbf{n})(e_0 + \epsilon\mathbf{n})}{1 - \epsilon^2}$$

and, for all $\mathbf{x} \in E$,

$$g(\epsilon\mathbf{n})(e_0 + \mathbf{x})g(\epsilon\mathbf{n})^{-1} = \frac{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}{1 - \alpha^2}\,(e_0 + \mathbf{x}'), \tag{3.13}$$

where

$$\alpha = \frac{2\epsilon}{1 + \epsilon^2}, \tag{3.14}$$

and

$$\mathbf{x}' = \frac{(1 - \alpha^2)\mathbf{x} + 2\alpha(1 + \alpha\,(\mathbf{n} \cdot \mathbf{x}))\mathbf{n}}{1 + \alpha^2 + 2\alpha\,(\mathbf{n} \cdot \mathbf{x})}. \tag{3.15}$$

In terms of the $\mathcal{C}$ realization given by Eqs (3.4),(3.10) the transformation $\mathbf{x} \mapsto \mathbf{x}'$ reads:

$$P(\mathbf{x}) \longmapsto P(\mathbf{x}') = \frac{P(\alpha\mathbf{n})\,P(\mathbf{x})\,P(\alpha\mathbf{n})}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}, \tag{3.16}$$

with $P(\alpha\mathbf{n}) \in \mathcal{C}$ given by

$$P(\alpha\mathbf{n}) = 1 + \alpha\mathbf{n}. \tag{3.17}$$

If $\mathbf{x} \in \bar{B}^{n+1}$ then $\mathbf{x}' \in \bar{B}^{n+1}$, and if $\mathbf{x} \in S^n$ then $\mathbf{x}' \in S^n$. The map $S^n \ni \mathbf{x} \mapsto \mathbf{x}' \in S^n$, given by Eq. (3.15), is conformal and it does not, in general, preserve the canonical, $SO(n+1)$–invariant, volume form dS of S^n. For every $\mathbf{x} \in S^n$ we have:

$$\frac{dS'}{dS}(\mathbf{x}) = \left(\frac{1 - \alpha^2}{1 + \alpha^2 + 2\alpha(\mathbf{n} \cdot \mathbf{x})}\right)^n. \tag{3.18}$$

If the map (3.15) is applied to the ball $B^{(n+1)}$ (rather than to its boundary S^n), and if dV denotes the standard Euclidean volume form of E^{n+1}, then

$$\frac{dV'}{dV} = \left(\frac{1-\alpha^2}{1+\alpha^2+2\alpha(\mathbf{n}\cdot\mathbf{x})}\right)^{n+2}. \tag{3.19}$$

Proof Since $(e_0+\epsilon\mathbf{n})^2 = (e_0-\epsilon\mathbf{n})^2 = \epsilon^2-1$, it follows that $g(\epsilon\mathbf{n})$ defined be Eq. (3.12) is in $Spin(1,n+1)$ and we have $g(\epsilon\mathbf{n})^{-1} = g(-\epsilon\mathbf{n}) = (e_0-\epsilon\mathbf{n})(e_0+\epsilon\mathbf{n})/(1-\epsilon^2)$. The formulae (3.13) and (3.15) follow then by a straightforward, though lengthy, calculation. Using the representation given by the formula (3.4) we get $g(\epsilon\mathbf{n})$ represented by:

$$A(g(\epsilon\mathbf{n})) = \begin{pmatrix} q(\alpha\mathbf{n}) & 0 \\ 0 & \pi(q(\alpha\mathbf{n})), \end{pmatrix}$$

with $q(\alpha\mathbf{n}) \in \mathcal{C}$ given by Eq. (3.17). Taking into account that $\pi(q(\alpha\mathbf{n})) = q(\alpha\mathbf{n})^{-1}$ we obtain Eq. (3.16). To prove Eq. (3.18) let us choose an orthonormal coordinate system in E so that the vector $\mathbf{n}$ has coordinates $x^1 = x^2 = \ldots = x^n = 0,\ x^{n+1} = 1$. Let us introduce spherical coordinates (cf. [21, p.240]) $\theta_1, \theta_2, \ldots, \theta_n,\ 0 \leq \theta_i \leq \pi,\ i = 2, \ldots, n,\ 0 \leq \theta_1 \leq 2\pi$, so that

$$\begin{array}{rcl}
x^1 & = & \sin(\theta_n)\sin(\theta_{n-1})\sin(\theta_{n-2})\ \ldots\ \sin(\theta_3)\sin(\theta_2)\sin(\theta_1), \\
x^2 & = & \sin(\theta_n)\sin(\theta_{n-1})\sin(\theta_{n-2})\ \ldots\ \sin(\theta_3)\sin(\theta_2)\cos(\theta_1), \\
x^3 & = & \sin(\theta_n)\sin(\theta_{n-1})\sin(\theta_{n-2})\ \ldots\ \sin(\theta_3)\cos(\theta_2), \\
 & \ldots & \\
x^n & = & \sin(\theta_n)\cos(\theta_{n-1}) \\
x^{n+1} & = & \cos(\theta_n).
\end{array}$$

The vector $\mathbf{n}$ has now spherical coordinates $\theta_1 = \theta_2 = \ldots\theta_n = 0$, and the transformation given by Eq. (3.13) takes the form

$$\left.\begin{array}{rcl}
\theta'_1 & = & \theta_1, \\
\ldots & \ldots & \\
\theta'_{n-1} & = & \theta_{n-1}, \\
\cos(\theta'_n) & = & \frac{(1-\alpha^2)\cos(\theta_n)+2\alpha(1+\alpha\cos(\theta_n))}{1+\alpha^2+2\alpha\cos(\theta_n)}.
\end{array}\right\} \tag{3.20}$$

The volume element dS for S^n in spherical coordinates is (cf. e.g. [21, p. 242])

$$dS = \sin^{n-1}(\theta_n)\sin^{n-2}(\theta_{n-1})\ldots\sin^2(\theta_3)\sin(\theta_2)d\theta_1\ldots d\theta_n. \tag{3.21}$$

From Eqs. (3.20) and (3.21) it follows that

$$\frac{dS'}{dS} = \frac{\sin^{n-1}(\theta'_n)d\theta'_n}{\sin^{n-1}(\theta_n)d\theta^n} = \frac{\sin^{n-2}(\theta'_n)d\cos(\theta'_n)}{\sin^{n-2}(\theta^n)d\cos(\theta_n)}.$$

Now, by a straightforward computation, using the last row of Eq. (3.20) we have

$$\frac{\sin^2(\theta'_n)}{\sin^2(\theta_n)} = \frac{1-\cos^2(\theta'_n)}{1-\cos^2(\theta_n)} = \frac{(1-\alpha^2)^2}{(1+\alpha^2+2\alpha\cos(\theta_n))^2} \tag{3.22}$$

and also

$$\frac{d\cos(\theta'_n)}{d\cos(\theta_n)} = \frac{(1-\alpha^2)^2}{(1+\alpha^2+2\alpha\cos(\theta_n))^2}. \tag{3.23}$$

Therefore, taking into account the fact that $\cos(\theta_n) = \mathbf{n}\cdot\mathbf{x}$, we get

$$\frac{dS'}{dS} = \frac{(1-\alpha^2)^n}{(1+\alpha^2+2\alpha\mathbf{n}\cdot\mathbf{x})^n} = \left(\frac{1-\alpha^2}{1+\alpha^2+2\alpha(\mathbf{n}\cdot\mathbf{x})}\right)^n, \tag{3.24}$$

as in (3.18). If the map (3.15) is applied to the $(n+1)-$dimensional open ball B^{n+1}, then, for $i = 1,\ldots,n$, $\partial x'^i/\partial x^i = (1-\alpha^2)/1+\alpha^2+2\alpha(\mathbf{n}\cdot\mathbf{x})$, $\partial x'^i/\partial x^{n+1} \neq 0$, and $\partial x'^{n+1}/\partial x^{n+1} = (1-\alpha^2)/1+\alpha^2+2\alpha(\mathbf{n}\cdot\mathbf{x})^2$, all other partial derivatives vanishing. Thus the Jacobi matrix $\partial x'/\partial x$ is triangular, and so its determinant is the product of the diagonal elements, as in Eq. (3.19), which completes the proof. ■

4 Iterated function systems of conformal maps

Let S be a set, let $\{w_i : i = 1,2,\ldots,N\}$ be a family of maps $w_i : S \longrightarrow S$, and let $p_i(x)$, $i = 1,2,\ldots,N$ be positive functions on S satisfying $\sum_{i=1}^N p_i(x) = 1$, $\forall x \in S$. The maps w_i and the functions $p_i(x)$ define what is called and *iterated function system (IFS) with place dependent probabilities* - cf. [22]. Starting with an initial point x_0 we select one of the transformations w_i with the probability distribution $p_i(x_0)$. If w_{i_1} is selected, we get the next point $x_1 = w_{i_1}(x_0)$, and we repeat the process again, selecting the next transformation w_{i_2}, according to the probability distribution $p_i(x_1)$. By iterating the process we produce a random sequence of integers $i_0, i_1, \ldots$ and a random sequence of points $x_k = w_{i_k}(x_{k-1}) \in S$, $k = 1,2,\ldots$.

In interesting cases the sequence x_k accumulates on an "attractor set" which has fractal properties. Instead of looking at the points of S we can take a dual look at the functions on S. Let $\mathcal{F}(S)$ be the set of all real–valued functions on S. $\mathcal{F}(S)$ is a vector space, and each transformation $w : S \to S$ induces a *linear* transformation $w^\star : \mathcal{F}(S) \to \mathcal{F}(S)$ defined by $(w^\star f)(x) = f(w(x))$, $x \in S$, $f \in \mathcal{F}(S)$.

4.1 Markov operator

Given an iterated function system $\{w_i,\, p_i(\,.\,)\}$ on S one naturally associates with it a linear *Markov operator* (sometimes called also the *transfer* operator) $T^* : \mathcal{F}(S) \to \mathcal{F}(S)$ defined by

$$(T^* f)(x) = \sum_{i=1}^{N} p_i(x)(w_i^* f)(x) = \sum_{i=1}^{N} p_i(x) f(w_i(x)). \tag{4.25}$$

There is a dual Markov operator T_*, acting on measures on S. Suppose S has a measurable structure, w_i and $p_i(\,.\,)$ are measurable, and let $\mathcal{F}(S)$ be the space of all bounded measurable functions on S. Let $\mathcal{M}(S)$ be the space of all finite measures on S. Then $T_* : \mathcal{M}(S) \to \mathcal{M}(S)$ is defined by duality: $(T_*\mu, f) = (\mu, T^* f)$, where $(\mu, f) \doteq \int f d\mu$. Since $T^*(1) = 1$, where $1(x) = 1$, $\forall x \in S$, we have that $\int dT_*\mu = \int d\mu$ and, in particular, T_* maps probabilistic measures into probabilistic measures. In many interesting cases the sequence of iterates $(T_*)^k \mu$ converges, in some appropriate topology, to a limit $\mu_\infty = \lim_{k\to\infty} (T_*)^k \mu$, that is independent of the initial measure μ, and which is the unique fixed point of T_*. The support set of μ_∞ is then the attractor set mentioned above.

Let μ_0 be a fixed, normalized measure on S, and assume that the maps w_i^{-1} map sets of measure μ zero into sets of measure μ zero. Then, for any finite k, the measure ${T_\star}^k \mu_0$ is continuous with respect to μ_0 and therefore can be written as

$${T_\star}^k \mu_0(\mathbf{r}) = f_k(\mathbf{r})\, \mu_0(\mathbf{r}). \tag{4.26}$$

The sequence of functions $f_k(\mathbf{r})$ gives a convenient graphic representation of the limit invariant measure. In our case, as it follows from the formula (4.26), the maps w_i are bijections, and the functions f_k can be computed explicitly via the following recurrence formula:

$$f_{k+1}(\mathbf{r}) = \sum_{i=1}^{N} p_i\left(w_i^{-1}(\mathbf{r})\right) \frac{d\mu_0\left(w_i^{-1}(\mathbf{r})\right)}{d\mu_0(\mathbf{r})}\, f_k\left(w_i^{-1}(\mathbf{r})\right). \tag{4.27}$$

4.2 Conformal maps

In this section the set S is either the sphere S^n, or the closed ball $\bar{B}^{n+1}$, and the maps w are of the form (3.15), and are determined by vectors $\alpha\mathbf{n} \in B^{(n+1)}$. Let us choose one α, $0 < \alpha < 1$, and N unit vectors $\mathbf{n}_i \in S^n$, so that we have N maps

$$w_i(\mathbf{x}) = \frac{(1-\alpha^2)\mathbf{x} + 2\alpha(1+\alpha(\mathbf{n}_i \cdot \mathbf{x}))\mathbf{n}_i}{1+\alpha^2+2\alpha(\mathbf{n}_i \cdot \mathbf{x})}, \tag{4.28}$$

as in Proposition 4. In our case we have an additional structure in the set S and in the maps w_i, namely the one stemming from the Clifford algebra realization. First of all to each $\mathbf{x} \in S^n$ we have associated the idempotent $P(\mathbf{x}) = \frac{1}{2}(1+\mathbf{x})$, and then we have a special class of functions on S, namely the functions of the form:

$$f_a(\mathbf{x}) = (P(\mathbf{x}), a),\ a \in \mathcal{C},\ \mathbf{x} \in \bar{B}(n+1). \tag{4.29}$$

We denote by $\mathcal{L}$ the vector space of these functions. Notice that functions in $\mathcal{L}$ separate the points $\mathbf{x} \in \bar{B}^{(n+1)}$. Indeed, for $\mathbf{x}, \mathbf{y} \in \bar{B}^{(n+1)}$ we have $f_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}/2$, thus our statement reduces to: for any two different vectors $\mathbf{x}_1, \mathbf{x}_2$ one can always find another vector $\mathbf{y}$ such that $\mathbf{x}_1 \cdot \mathbf{y} \neq \mathbf{x}_2 \cdot \mathbf{y}$, which is evident. [8]

Proposition 5. *With the notation as in the beginning of this section, let* $0 < \alpha < 1$, $\mathbf{n}_i \in S^n$, $i = 1, 2, \ldots N$ *and* w_i *as in Eq. (4.28). Suppose that*

1)
$$\sum_{i=1}^{N} \mathbf{n}_i = 0, \tag{4.30}$$

2)
$$p_i(\mathbf{x}) = \frac{1+\alpha^2+2\alpha(\mathbf{n}_i \cdot \mathbf{x})}{Z(\alpha)}, \tag{4.31}$$

where

$$Z(\alpha) = \sum_{i=1}^{N}(1+\alpha^2+2\alpha(\mathbf{n}_i \cdot \mathbf{x})) = N(1+\alpha^2),$$

[8] The space $\mathcal{L}$ is $(n+2)$–dimensional, as it is clear that $f_a(\mathbf{x}) \equiv 0$ for $a \in C_p \subset \mathcal{C},\ p > 1$.

then the Markov operator T^* of the iterated function system $\{(w_i, p_i)\}$ maps the space $\mathcal{L}$ into itself: $T^* : f_a \mapsto f_V(a)$, where

$$V(a) = \frac{1}{N(1+\alpha^2)} \sum_{i=1}^{N} P(\alpha \mathbf{n}_i)\, a\, P(\alpha \mathbf{n}_i). \tag{4.32}$$

Proof: From Eq. (3.13) it follows that if $\sum_i \mathbf{n}_i = 0$, then $Z \doteq \sum_i (1+\alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})) = N(1+\alpha^2)/(1-\alpha^2)$ is a constant, independent of $\mathbf{x}$. From the very definition of the Markov operator, as well as from Eqs (4.29),(3.16) it follows then that

$$\begin{aligned}
(T^* f_a)(\mathbf{x}) &= \sum_i p_i(\mathbf{x}) f_a(w_i(\mathbf{x})) = \sum_i p_i(\mathbf{x}) \Phi(a\, P(w_i(\mathbf{x}))) \\
&= \sum_i p_i(\mathbf{x}) \Phi\left(a \frac{1-\alpha^2}{(1+\alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x}))} P(\alpha \mathbf{n}_i) P(\mathbf{x}) P(\alpha \mathbf{n}_i) \right) \\
&= \sum_i p_i(\mathbf{x}) \frac{(1-\alpha^2)}{1+\alpha^2 + 2\alpha(\mathbf{n}_i \cdot \mathbf{x})} \Phi\left(P(\alpha \mathbf{n}_i) a P(\alpha \mathbf{n}_i) P(\mathbf{x}) \right) \\
&= \frac{1}{Z(\alpha)} \sum_i \Phi\left(P(\alpha \mathbf{n}_i) a P(\alpha \mathbf{n}_i) P(\mathbf{x}) \right) = f_{V(a)}(\mathbf{x}).
\end{aligned}$$

■

The Markov operator T_* acts on measures, while its dual T^* acts on functions on S. Every probabilistic measure μ on S determines an algebra element $P(\mu)$ defined by:

$$P(\mu) = \int_S P(\mathbf{x})\, d\mu(\mathbf{x}) = 1 + \int_S \mathbf{x}\, d\mu(\mathbf{x}) = P\left(\int_S \mathbf{x}\, d\mu(\mathbf{x}) \right), \tag{4.33}$$

so that automatically $\Phi(P(\mu)) = 1$. $P(\mu)$ is an idempotent if and only if μ is concentrated at just one point on the boundary S^n. In general there are infinitely many measures μ giving rise to the same algebra element $P(\mu)$. The process of integration on one hand leads to simplification (linearization) but, on the other hand, it also leads to the loss of information.

Corollary 1. *Under the assumptions 1) and 2) of Proposition 5, if μ_1 and μ_2 are two probabilistic measures on S such that $P(\mu_1) = P(\mu_2) = P$, then $P(T_*\mu_1) = P(T_*\mu_2) = V(P)$, where $V(P)$ is given by the formula (4.32), with a replaced by P.*

Proof: Because functions $f_a,\ a \in \mathcal{C}$ separate the elements of $\mathcal{C}$, it is enough to show that $f_a(P(T_*\mu)) = f_a(V(P(\mu)))$ for all $a \in \mathcal{C}$. Now, from the very definition of the functions $f_a,\ f_a(\mathbf{x}) = \Phi(aP(\mathbf{x}))$, and from the linearity of the trace functional Φ, it follows that $(f_a, \mu) \doteq \int f_a(\mathbf{x})d\mu(x) = \Phi(aP(\mu))$, and so $f_a(V(P(\mu)) = \Phi(aV(P(\mu))) = \Phi(V(a)P(\mu)) = f_V(a)(P(\mu)) = f_a(P(T_*\mu))$. QED ■

Corollary 2. *Under the assumptions 1) and 2) of Proposition 5, the Markov operator recurrence formula (4.27) is explicitly given by*

$$f_{k+1}(\mathbf{r}) = \frac{(1-\alpha^2)^{n+2}}{N(1+\alpha^2)} \sum_{i=1}^{N} \frac{f_k\left(w_i^{-1}(\mathbf{r})\right)}{\left(1+\alpha^2-2\alpha(\mathbf{n}_i\cdot\mathbf{x})\right)^{n+1}}, \tag{4.34}$$

where

$$w_i^{-1}(\mathbf{r}) = \frac{(1-\alpha^2)\mathbf{r} - 2\alpha(1-\alpha(\mathbf{n}_i\cdot\mathbf{r}))\mathbf{n}_i}{1+\alpha^2-2\alpha(\mathbf{n}_i\cdot\mathbf{r})}. \tag{4.35}$$

Proof: Follow directly by a somewhat lengthy calculation using the Eqs. (4.27),(4.28),(4.31), and (3.19). QED ■

Note: Iterated function systems for mixed states have been discussed by Łozinski et al. in Ref. [23], while Słomczynski [24] discussed Markov operators and dynamical entropy for general IFS–s on state spaces. In these references the probability distributions assigned to the maps were generic rather than derived geometrically, as is the case in this paper.

5 Examples

5.1 S^1 – Polygon

As the first example we consider the circle S^1, and unit vectors $\mathbf{n}_i$ pointing to the vertices of a regular polygon. For an illustration we choose the pentagon. Fig. 1 shows the plot of $\log_1 0(f_7 + 1.0)$, the 7–th iteration of the Markov operator – see Eq. (4.34), for $\alpha = 0.58$.

5.2 S^2

S^2, the Riemann sphere, is the same as the complex projective line $P^1(\mathbb{C})$ - the space of pure quantum states of the simplest non–trivial quantum system, namely spin 1/2. Examples of quantum fractals on S^2, based on

Platonic solids, has been given elsewhere (cf. [28], and references therein). Here we give just one example, namely the octahedral quantum fractal. Fig. 2 shows the 7–th power of the Markov operator: $\log_{10}(f_7 + 1)$, - cf. Eq. (4.34) for $\alpha = 0.5$, plotted on the projection of the upper hemisphere of S^2. The emergence of circles on the plot is rather surprising and not well understood.[9]

5.3 S^3 – regular polytopes

There are six regular polytopes in four dimensions: self–dual pentachoron (or 4 simplex), 16 cell (or cross–polytope, or hexadecochoron), dual to it 8 cell (or hypercube or tesseract), self–dual 24 cell (or icositetrachoron), 600 cell (or hexacosichoron), and its dual 120 cell (or hecatonicosachoron) - cf. Fig. 3 and Fig. 8. In our examples of four dimensional quantum fractals we skip the first one. The pentachoron (the four dimensional equivalent of the tetrahedron) leads to rather trivial and uninteresting fractal pattern.

5.4 S^3 – 16 cell.

Quaternions of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{Z}$ form the so called *Lipschitz* ring. The unit quaternions of this ring form a group of order 8 - the binary dihedral group $D4$. Its eight elements, $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ form the four-dimensional regular polytope, the so called *cross–polytope*, with Schläfli symbol $\{3, 3, 4\}$. It has 16 tetrahedral cells, each of its 24 edges belongs to 4 cells.

Visualization of quantum fractals that live in four dimensions is difficult. Here we generate 10,000,000 points of the iterated function system described in Sec 4.2, with $\mathbf{n}_i$ being the 8 vertices of the 16 cell, $\alpha = 0.5$, and with probabilities given by Eq. (4.31). We plot the three dimensional projections of those 16742 points which fall into the slice of S^3 with the fourth coordinate $0.5 < x^4 < 0.51$ - see Fig. 4.

This pattern, generated by the IFS of conformal maps with place--dependent probabilities should be compared with the plot of the fourth approximation to the density of the limiting invariant measure - Fig. 4. Due to the recursive nature of the formula Eq. (4.34) the computation time of f_k grows exponentially with k. With each level new details appear in the

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[9] The algorithm for generating conformal quantum fractals on S^2 has been included in the CLUCalc software by Christian Perwass. A video zooming on a quantum fractal based on the regular octahedron, $\alpha = 0.42$, can be seen on the CLUCalc home page: `http://www.clucalc.info/`

graph, at the same time the probability peaks get higher (as in Fig. 6). To present more details in the graph, we are plotting $\log_{10}(f_4(\mathbf{r})+1)$, rather than the function $f_4(\mathbf{r})$ itself. Notice that for each k, the integral of $f_k(\mathbf{r})$ over the sphere S^3, with the natural $SO(4)$ invariant measure , is constant and equal to the volume of S^3.

5.5 S^3 − 8 cell.

Dual to the 16 cell is the 8 cell, also known as *cross polyhedron hypercube*, or *tesseract*. Its 16 vertices are the unit quaternions $\frac{1}{2}(\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k})$. Its Schläfli symbol is $\{4,3,3\}$, which means that its cells are $\{4,3\}$ - that is cubes, each face belongs to 2 cells, and each edge belongs to 3 cells. The hypercube is built of two 3 dimensional cubes, their edges being connected along the fourth coordinate. The projection of the hypercube is shown in Fig. 3.

We choose 16 unit vectors $\mathbf{n}_i$ pointing to the vertices of the hypercube. Fig. 5 shows the plot of f_5, the 5–th iteration of the Markov operator (given by Eq. (4.34), for $\alpha = 0.60$, restricted to the section $x^3 = 0.8$, projected onto (x^1, x^2) plane.

5.6 S^3 − 24 cell.

Quaternions of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a,b,c,d \in \mathbb{Z}$ or $a,b,c,d \in \mathbb{Z} + \frac{1}{2}$ form a ring, called the *Hurwitz ring*. Its additive group is the F_4 lattice. The unite quaternions of this ring form a group, the *binary tetrahedral group* T_24, isomorphic to the group $SL(2,3)$ - with generators the same as for $SL(2,5)$, - cf. Eq (8.41), except that the multiplications are carried in $\mathbb{Z}_3$. 24 cell has Schläfli symbol $\{3,4,3\}$, which means that its 24 cells are octahedrons, with each edge belonging to three cells [29, p. 68]. Each of its 16 vertices is common to 6 cells - cf. Fig. 3. Fig. 6 shows the plots of $\log(f_k)+1)$ for $k = 2,3,4$, for $x^4 = 0.5$, and $\alpha = 0.6$. With each power of the Markov operator more details of the limit measure appear.

5.7 S^3 − 600 cell.

Here we provide an example of a quantum fractal on S^3, based on the regular polytope in four dimensions, namely the 600 cell, with Schläfli symbol $\{3,3,5\}$. The vertices of the 600 cell are given in the Appendix 1b. (c.f. also [29, p. 74–75].) Fig. 3 shows a two dimensional projection of the 600 cell as viewed from the direction of the center of one of its cells, while Fig. 8 (top) shows the more perfect (all 120 vertices can bee seen) Coxeter's projection. The inner ring, consisting of 30 vertices is on the torus. We show the functions $\log_1 0(f_1+1)$ and $log_1 0(f_2+1)$ plotted at the surface of this torus. The 30 highest peaks that can be seen on the bottom plots are located at the vertices.

5.8 S^3 − 120 cell.

The last example is the 120 cell, with 600 vertices. Fig. 9 (top) shows a particular projection of this polytope, with one of its 120 octahedral cells plotted in bold. Below is the plot of $\log_1 0(f_2+1)$, for $\alpha = 0.9$, at the upper hemisphere circumscribing this cell.

6 Summary and conclusions

In the standard formulation of the quantum theory the imaginary unit i plays an important yet somewhat mysterious role: it appears in front of the Planck constant $\hbar$, and provides a one–to–one formal correspondence between hermitian "observables" and anti–hermitian generators of one–parameter groups of unitary transformations. In particular it is necessary in order to write the time evolution equation for the wave function, with the energy operator (the Hamiltonian) defining the evolution. But the imaginary "i" is not needed for quantum jumps. In a theory where quantum jumps are the driving force of the evolution, the real algebra structure, with a real trace functional suffices. In the present paper we have studied the simplest case of real Clifford algebras of Euclidean spaces and demonstrated that from the algebra and from the geometry a natural family of iterated function systems of conformal maps leads to fractal structures and pattern formation on spheres S^n. In this way we open a way towards algebraic generalizations of quantum theory that are based on discrete, algebraic structure, as expressed in the late Einstein's vision quoted in the Introduction.

Among the open problems we would like to point out particularly the following ones.

6.1 Existence and uniqueness of the invariant measure

While numerical simulations (see the next section), suggest that for the class of iterated function systems discussed in this paper, the attractor set and the invariant measure exists and is unique, we are not able to provide a mathematical proof. Even if the spheres S^n and balls B^{n+1} are compact, the Möbius transformations of these spheres are non-contractive. The question of existence and uniqueness of invariant measures for non-contractive iterated function systems has been discussed in the mathematical literature [25, 26, 27], yet none of the sufficient conditions seems to be easily applicable to our case. Apanasov has a whole book devoted to conformal maps, yet we find that his criteria, esp. Theorem 4.16 of Ref. [27], are abstract and difficult to apply. Therefore the problem of existence and uniqueness of the invariant measure for IFS–s discussed in the present paper remains open at this time.

6.2 Fractal dimension as a function of the parameter ϵ.

Anticipating a positive answer to the above problem, the next important question is the exact nature of the fractal attractor as a function of the parameter ϵ. The numerical simulations seems to suggest that the fractal dimension of the attractor of our IFSs on S^n decreases, starting from n, for $\epsilon = 0$. Yet our attempt to determine its behavior, even for the simplest case of S^1, met an obstacle. We tried to calculate the correlation dimension for the pentagon case, described in Example 1. To this end we generated 10,000,000 points, using the algorithm of Sec. 4, and plotted, on the log–log scale the function $C(N, r)$, where r is the distance between two points, and N is the number of pairs of points within this distance. More precisely, the correlation dimension D is defined as

$$D = \lim_{r \to 0} \log(C(r))/\log(r),$$

where

$$C(r) = \frac{1}{N^2} \lim_{N \to \infty} \sum_{i,j}^{N} \Theta(|r - |x_i - x_j|),$$

Θ being the unit step function. For the standard Cantor set the correlation dimension algorithm gives the correct fractal dimension, namely $D = 0.63 \approx \log(2)/\log(3)$. For the pentagon, with $\epsilon = 0.58$, (cf. Fig. 1) we get a reasonable straight line with the slope $D \approx 0.9$, but with $\epsilon = 0.925$, when the expected fractal dimension should be close to zero, we get a staircase. It is not clear whether this is due to numerical artifacts, or is it a pointer towards the possible multifractality of quantum fractals for high values of ϵ.

7 Appendix 1a - Hamilton's Icosian Calculus

Hamilton's "Icosian Calculus" dates back to his communication to the Proc. Roy. Irish Acad. of November 10, 1856 [30, p.609], followed by several papers, the last one in 1863. According to the contemporary terminology Hamilton proposes a particular presentation of the alternating group $A5$ - the symmetry group of the icosahedron.

Account of the Icosian Calculus
Communicated 10 November 1856.

$$Proc. Roy. Irish Acad. vol. vi (1858), pp. 415-16.$$

Sir William Rowan Hamilton read a Paper on a new System of Roots of Unity, and of operations therewith connected: to which system of symbols and operations, in consequence of the geometrical character of some of their leading interpretations, he is disposed to give the name of the "ICOSIAN CALCULUS". This Calculus agrees with that of the Quaternions, in three important respects: namely, 1st that its three chief symbols ι, κ, λ are (as above suggested) roots of unity, as i, j, k are certain fourth roots thereof: 2nd, that these new roots obey the associative law of multiplication; and 3rd, that they are not subject to the commutative law, or that their places as factors must not in general be altered in a product. And it differs from the Quaternion Calculus, 1st, by involving roots with different exponents; and 2nd by not requiring (so far as yet appears) the distributive property of multiplication. In fact, $+$ and $-$, in these new calculations, enter only as connecting exponents, and not as connecting terms: indeed, no terms, or in other words, no polynomes, nor even binomes, have hitherto presented themselves, in these late researches of the author. As regards the exponents of the new roots, it may be mentioned that in the principal system - for the new Calculus involves a family of systems-there are adopted the equations,

$$1 = \iota^2 = \kappa^3 = \lambda^5, \ \lambda = \iota\kappa; \quad (A)$$

so that we deal, in it, with a new square root, cube root, and fifth root, of positive unity; the latter root being the product of the two former, when taken in the order assigned, but not in the opposite order. From these simple assumptions (A), a long train of consistent calculations opens itself out, for every result of which there is found a corresponding geometrical interpretation, in the theory of two of the celebrated solids of antiquity, alluded to with interest by Plato in the Timaeus; namely the Icosahedron, and the Dodecahedron: whereof the angles may now be unequal. By making $\lambda^4 = 1$, the author obtains other symbolical results, which are interpreted by the Octahedron and the Hexahedron. The Pyramid is, in this theory, almost too simple to be interesting: but it is dealt with by the assumption, $\lambda^3 = 1$, the other equations (A) being untouched. As one fundamental result of those equations (A), which may serve as a slight specimen of the rest, it is found that if we make $\iota\kappa^2 = \mu$, we shall have

$$\mu^5 = 1,\ \mu = \lambda\iota\lambda,\ \lambda = \mu\iota\mu;$$

so that this new fifth root mu has relations of perfect reciprocity with the former fifth root lambda. But there exist more general results, including this, and others, on which Sir W. R. H. hopes to be allowed to make a future communication to the Academy: as also on some applications of the principles already stated, or alluded to, which appear to be in some degree interesting.

Today we know that the group $A5$ is simple, therefore it has no non–trivial invariant subgroups, therefore Hamilton's original comments about models that assume $\lambda^3 = 1$ or $\lambda^4 = 1$ are contradictory.

8 Appendix 1b - The Binary Icosahedral Group

Putting $R = \iota$, $S = \kappa$, $T = \lambda^4$, we can equivalently write Hamilton's equations (A) (Sec. 7) as

$$R^2 = S^3 = T^5 = RST = 1. \tag{8.36}$$

Removing the last equality we get the code for the *binary icosahedral group:*

$$R^2 = S^3 = T^5 = RST. \tag{8.37}$$

It is evident from the definition that $Z = RST$ is a central element of the group, and it can be shown [31, p. 69 and references therein] that Z is of order 2: $Z^2 = 1$. This group if order 120, denoted as $2.A5$, and it is a double cover of the icosahedral group $A5$. The group has a particularly simple representation in terms of the quaternions. Let

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots, \quad \Phi = \frac{-1+\sqrt{5}}{2} = \phi^{-1} = 0.61803\ldots, \quad (8.38)$$

be the Golden Ratio and its inverse, respectively. Consider the group G consisting of 120 elements given by Table 1 below:

Table 1: 120 vertices of the 600 cell

$2 \times 4 = 8$	elements of the form $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0),$ $(0, 0, \pm 1, 0), (\pm 0, 0, 0, \pm 1)$
$2^4 = 16$	elements of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$
$3! \times 2^3 = 96$	elements that are even permutations of elements of the form $\frac{1}{2}(\pm\phi, \pm 1, \pm\Phi, 0)$.

These 120 elements form a group of unit icossians (cf. Appendix 1a) that is a finite subgroup of the group $Spin(3)$. For generators R, S we can take, for instance[10],

$$S_1 = \frac{1}{2}(1 - \Phi i - \phi k), \quad T_1 = \frac{1}{2}(\Phi - i - \phi j), \quad R_1 = S_1 T_1 = -i, \quad (8.39)$$

or an inequivalent set

$$S_2 = \frac{1}{2}(1 + \phi i + \Phi j), \quad T_2 = \frac{1}{2}(-\phi - i - \Phi k). \quad R_2 = S_2 T_2 = -i. \quad (8.40)$$

In both cases we have $RST = -1$, but the two sets of generators are geometrically inequivalent (they are related by an *outer* automorphism of G), the angle between S_1 and T_1 is $\pi/5$ while the angle between S_2 and T_2 is $3\pi/5$.

[10]One can check that there are 120 possible choices of triples of quaternionic generators R, S, T satisfying Eq. (8.37).

The binary icosahedral group is isomorphic to $SL(2,5)$, the group of unimodular 2×2 matrices over the field Z_5, as can be seen by taking for the generators R, S, T the matrices:

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{8.41}$$

Fig. 3 shows the vertices of the 600 cell as viewed from the direction of the center of one of its cells. There is another realization of the 600 cell as a polytope, due to Coxeter [32, p.247], where all of the 120 vertices are organized on four different tori within the sphere S^3. Let

$$a = \sqrt{(1+3^{-1/2}5^{-1/4}\phi^{3/2})/2} \approx 0.947274,$$

$$b = \sqrt{(1+3^{-1/2}5^{-1/4}\phi^{-3/2})/2} \approx 0.770582,$$

$$c = \sqrt{(1-3^{-1/2}5^{-1/4}\phi^{-3/2})/2} \approx 0.637341,$$

$$d = \sqrt{(1-3^{-1/2}5^{-1/4}\phi^{3/2})/2} \approx 0.320426,$$

let $\theta = \pi/30$, and let the four families, each of 30 vertices, be given by:

$$\begin{aligned} a[k] &= \{a\cos(k\theta),\ a\sin(k\theta), \quad d\cos(11k\theta), \quad d\sin(11k\theta)\}, \\ b[k] &= \{d\cos(k\theta),\ d\sin(k\theta), \quad -a\cos(11k\theta), \quad -a\sin(11k\theta)\}, \end{aligned} \tag{8.42}$$

where

$$k = 0, k < 60, k = k+2,$$

and

$$\begin{aligned} a[k] &= \{b\cos(k\theta),\ b\sin(k\theta), \quad c\cos(11k\theta), \quad c\sin(11k\theta)\}, \\ b[k] &= \{c\cos(k\theta),\ c\sin(k\theta), \quad -b\cos(11k\theta), \quad -b\sin(11k\theta)\}, \end{aligned} \tag{8.43}$$

where

$$k = 1, k \leq 60, k = k+2.$$

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9 Figures

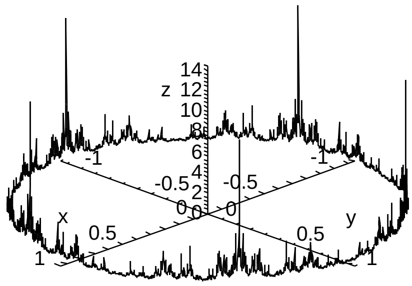


Figure 1: Pentagon. 7–th power of the Markov operator applied to $f \equiv 1$.

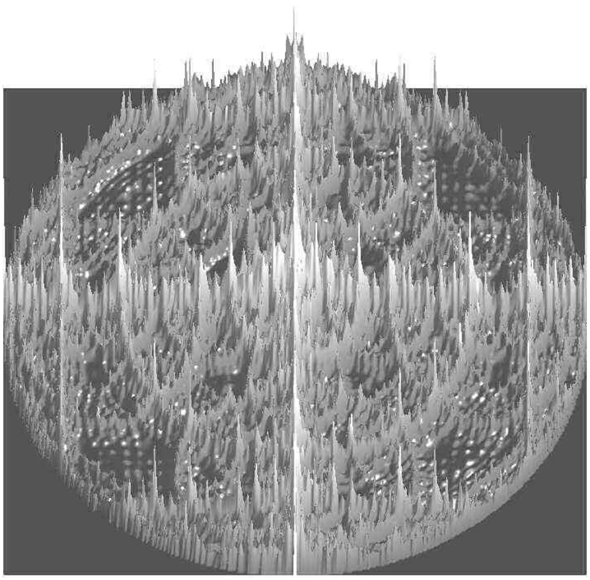


Figure 2: Octahedron – {3,4}. 7–th power of the Markov operator, $\alpha = 0.5$.

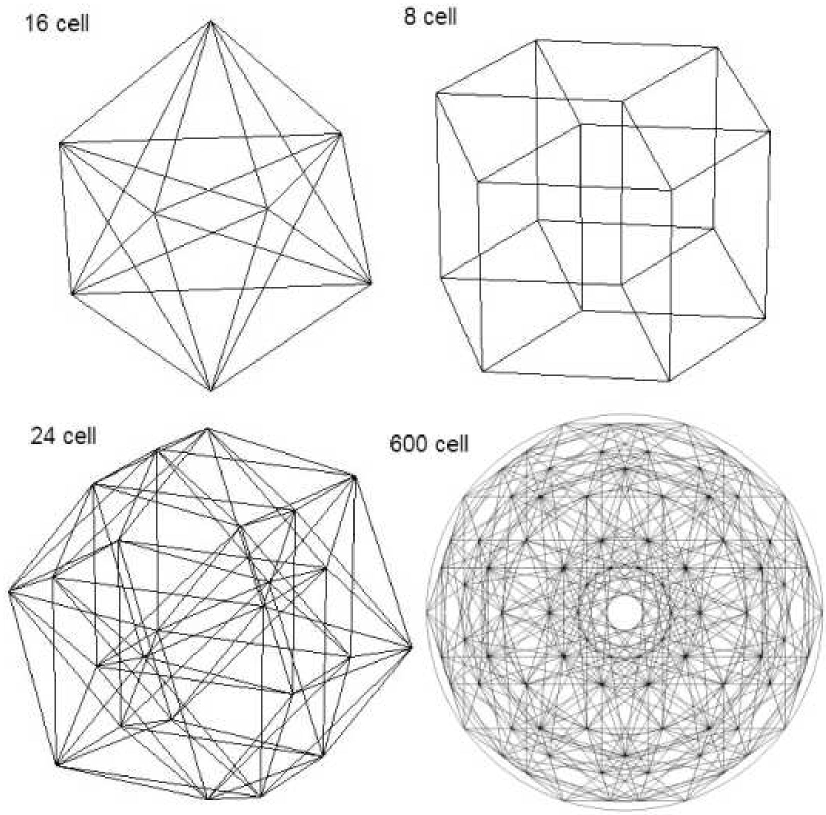


Figure 3: a) 16 cell - {3,3,3}. 8 vertices, 24 edges, 32 triangular faces, 16 tetrahedral cells. b) 8 cell or Hypercube - {4,3,3}. 16 vertices, 32 edges, 24 square faces, 8 cubic cells. c) 24 cell - {3,4,3}. 24 vertices, 96 edges, 96 triangular faces, 24 octahedral cells. d) 600 cell - {3,3,5}. 120 vertices, 720 edges, 1200 triangular faces, 600 tetrahedral cells. The graphics was generated by choosing the tetrahedral cell with vertices $t_0 = (1, 0, 0, 0)$, $t_1 = (\phi, \Phi, 0, 1)/2$, $t_2 = (\phi, 0, 1, \Phi)/2$, $t_3 = (\phi, 1, \Phi, 0)/2$, and choosing the unit vector f_1 in the direction of the center of this cell $(t_0+t_1+t_2+t_3)/4$. The second unit vector f_1 was chosen in the direction of $f_0 * t_1$, (the quaternionic product). Then the frame $(f_0, f_1, f_2 = (0, 0, 1, 0), f_3 = (0, 0, 0, 1))$ was orthonormalized to (e_0, e_1, e_2, e_3) via Gram-Schmidt procedure, and the 720 edges of the 600 cell have been projected onto (e_2, e_3) plane.

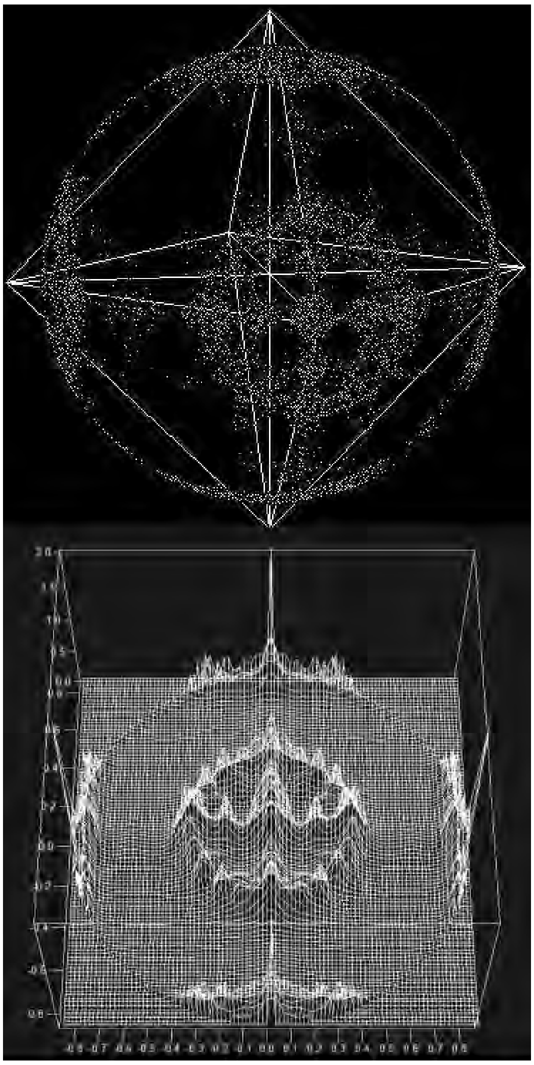


Figure 4: 16 cell – {3,3,4}. Generated 10,000,000 random points of the IFS system of conformal maps with $\alpha = 0.5$. Plotted are 16742 points whose fourth coordinate is in the slice $0.5 < x^4 < 0.51$. The picture is superimposed on the projection of the edges of the 16 cell. Below: Plotted the fourth power of the Markov operator, more precisely of the function $\log_{10}(f_4(\mathbf{r})+1)$, with f_4 function defined in Eq.(4.34), calculated for $\alpha = 0.5$ and $x^4 = 0.5$.

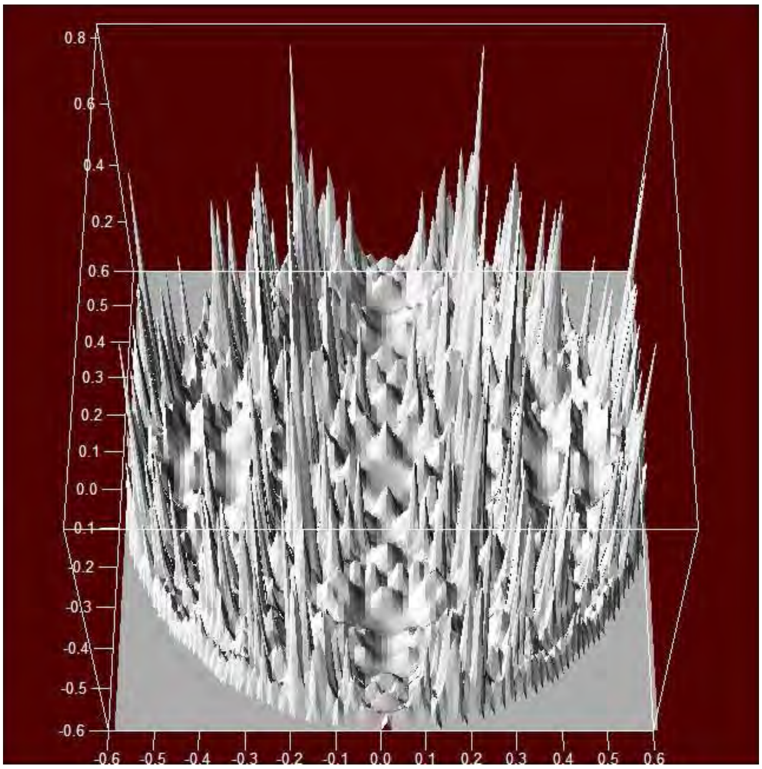


Figure 5: Hypercube – {4,3,3}. 5 th power of the Markov operator, Eq. (4.34), with $\alpha = 0.6$, computed at the section $x^4 = 0.8$. Plotted is the $\log_{10}((f_5) + 1)$.

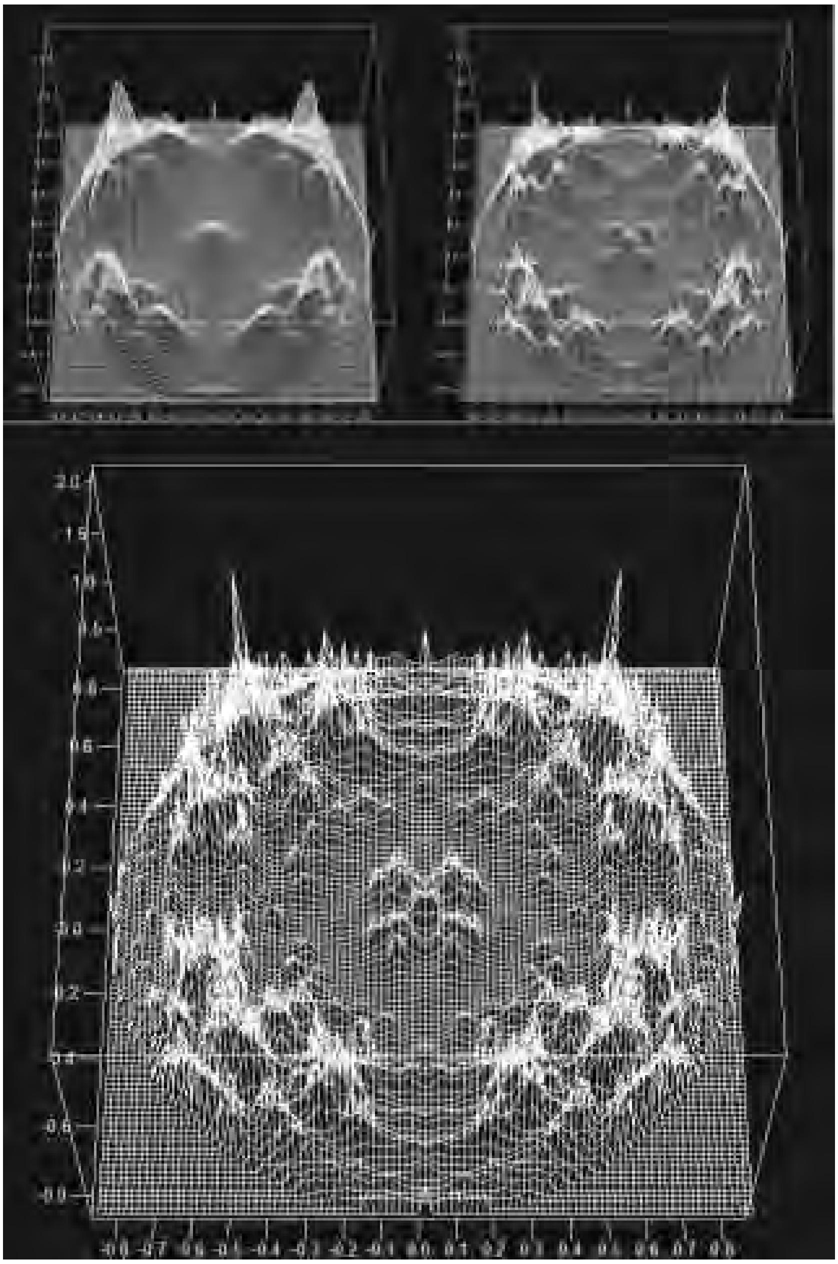
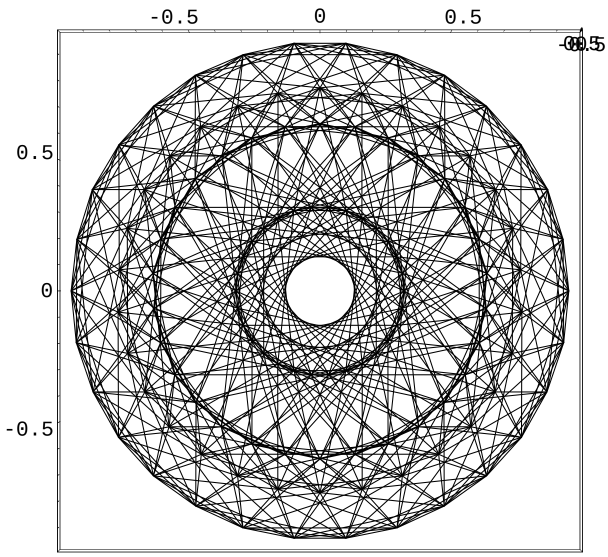


Figure 6: 24 cell – {3,4,3}. Markov operator levels 2,3 and 4, for $\alpha = 0.6$, plotted at $x^4 = 0.5$.



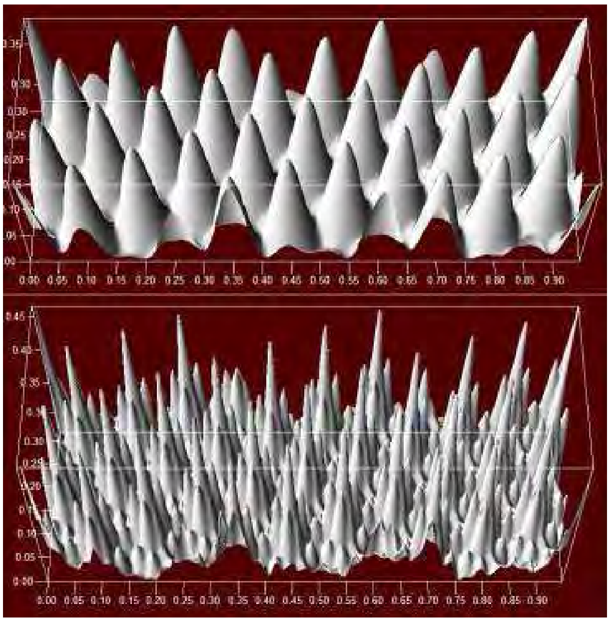


Figure 8: 600 cell - {3,3,5}. Top: Coxeter's projection. Below 1st and 2nd powers of the Markov operator, for $\alpha = 0.6$ plotted at the surface of the most inner torus.

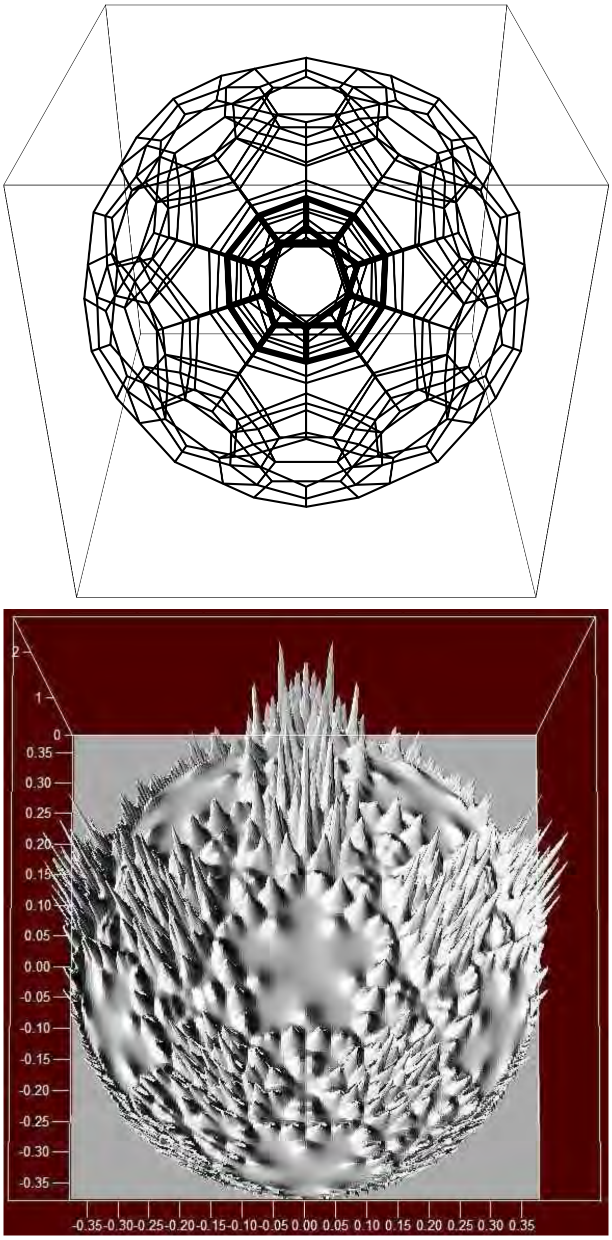


Figure 9: 120 cell – {5,3,3}. 600 vertices, 1200 edges of length $(1-\phi)/\sqrt(2)$, 720 pentagonal faces, 120 dodecahedral cells. One of its dodecahedral cells in bold. Below the 2nd power of the Markov operator, for $\alpha = 0.9$, plotted at the upper hemisphere of this particular cell.

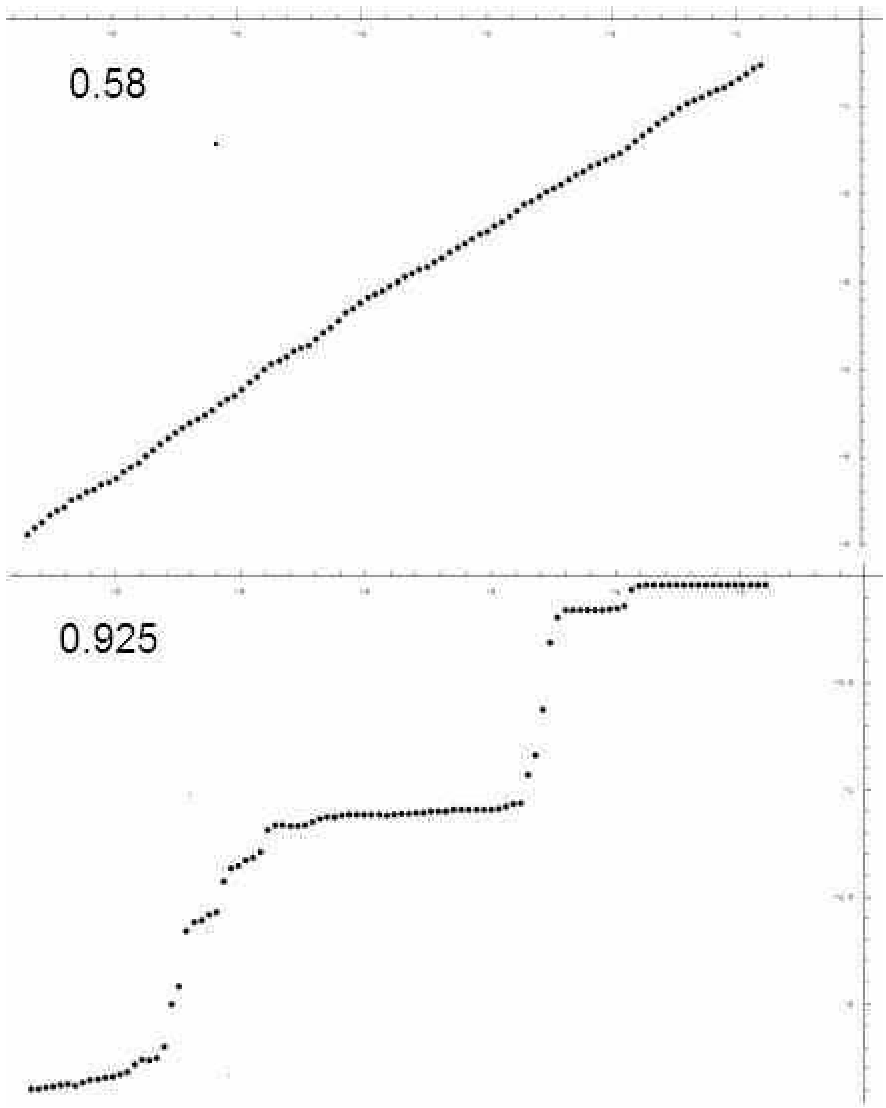


Figure 10: Correlation dimension plots for the pentagon, for $\alpha = 0.58$, and $\alpha = 0.925$.

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