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LOGICS GENERATED BY CAUSALITY STRUCTURES. COVARIANT REPRESENTATIONS OF THE GALILEAN LOGIC

W. Cegła and A. Z. Jadczyk

Institute of Theoretical Physics, University of Wrocław, 50-205, Wrocław, Poland

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We consider the causal structure of space-time in the logic approach. The general form of covariant representations of the Galilean logic, which correspond to a localizable Galilean system, is found.

1. Introduction

The goal of this work is to start a systematic investigation of the causal structure of space-time. This structure reflects itself in a structure of the algebra of observables of a quantum system via the correspondence $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ between space-time regions $\mathcal{O}$ and the algebras of observables related to $\mathcal{O}$ (see Haag and Kastler [5]). For an elementary particle we can take $\mathcal{A}(\mathcal{O})$ to be a von Neumann algebra generated by projections localizing the particle in subregions of $\mathcal{O}$. Unfortunately, by the theorem of Borchers [3], it is impossible to reconcile such a localization of a relativistic particle with the positivity of the energy. However, relativistic wave equations (finite or infinite component ones) are of great importance in spite of the fact that they admit negative-energy solutions. On the other hand, there is a problem (both relativistic and non-relativistic) of the satisfactory space-time description of unstable particles. Such an unstability induces in space-time a causality structure which is stronger than in the stable case. This is why we try to formulate an abstract theory of the causality. The Galilean causality is then discussed in some details. A general form (not the most general) of a covariant representation of the persistent Galilean logic is found to correspond to a localization of a distinguished point of a free composed system (infinite-dimensional Galilean wave equations). A different approach to similar problems can be found in the work of Barut and Malin [2].

2. Causality spaces

DEFINITION. Let X be a set. A *causal structure* (or, in short, *causality*) in X is a structure defined by a distinguished covering $\mathscr{G}$ of X by non-empty subsets. The sets from $\mathscr{G}$ are called *(causal) paths* of the causal structure defined in X by $\mathscr{G}$. A *causality space* is a set equipped with a causal structure. If a point $x \in X$ is an element of a path Γ, Γ is said to be *passing through* x. Two points x and y in X are said to be *causally related* if there is some path Γ passing through both of them.

DEFINITION. Let $(X, \mathscr{G})$ be a causality space. A subset $A \subset X$ is said to be *causally closed* if A is a complement of a union of some family of paths. The family of all causally closed sets is denoted by $\tilde{\mathscr{L}}$.

DEFINITION. We say that a point x is *(causally) controlled* by a set A in causality space X if each path passing through x passes also through A. The set of all points controlled by A is said to be a *(causal) closure* of A and denoted by $D(A)$.

We note that $D(A)$ is the smallest causally closed set containing A:

$$D(A) = (\bigcup\{\Gamma \in \mathscr{G};\ \Gamma \cap A = \emptyset\})^c$$

(compare Penrose [6], formula 9.8). Clearly, $A \subset B$ implies $D(A) \subset D(B)$, and A is causally closed if and only if it coincides with its causal closure.

DEFINITION. We say that a subset A *is controlled* by B if every point of A is controlled by B. In this case we write

$$A \propto B.$$

In other words, $A \propto B$ means that every path passing through A is necessarily passing through B or, equivalently, that $A \subset D(B)$. The relation $\propto$ is transitive and reflexive. The relation $A \propto\!\!\!\propto B$, denoting $A \propto B$ or $B \propto A$, is an equivalence relation in 2^X. Clearly, $A \propto\!\!\!\propto B$ means also $D(A) = D(B)$.

The two sets A and B are said to be *causally equivalent* if they have the same causal closure. If $[A]$ stands for the causal equivalence class of A, then the projection $A \to [A]$ admits a natural section $[A] \to D(A)$. For each A, $D(A)$ is the largest set causally equivalent to A.

DEFINITION. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two causalities in a set X. If every path of $\mathscr{G}_1$ is a path of $\mathscr{G}_2$, then $\mathscr{G}_1$ is said to be *weaker* than $\mathscr{G}_2$, and $\mathscr{G}_2$ is said to be *stronger* than $\mathscr{G}_1$.

If $\mathscr{G}_2$ is stronger than $\mathscr{G}_1$, then every two points causally related in $\mathscr{G}_1$ are causally related in $\mathscr{G}_2$. For every $A \subset X$ the causal closure of A in $\mathscr{G}_1$ contains the causal closure of A in $\mathscr{G}_2$. Every set causally closed in $\mathscr{G}_1$ is causally closed in $\mathscr{G}_2$ (if $\mathscr{G}_1 \subset \mathscr{G}_2$, then $\tilde{\mathscr{L}}_1 \subset \tilde{\mathscr{L}}_2$).

DEFINITION. A subset A of X is said to be *causally connected* if every two points of A are causally related. We say that A is *causal* if there exists a causal path Γ which contains A. A causality $\mathscr{G}$ in X is said to be *complete* if every causally connected set is causal; it is said to be *persistent* if each of its paths is a maximal causally connected set. Every causally connected (causal) set with respect to $\mathscr{G}$ is causally connected (causal) in any causality $\mathscr{G}'$ stronger than $\mathscr{G}$.

PROPOSITION 1. *Let $\mathscr{G}$ be a causality in X. There exists a causality $\tilde{\mathscr{G}}$ such that*

(i) *$\tilde{\mathscr{G}}$ is stronger than $\mathscr{G}$,*

(ii) *$\tilde{\mathscr{G}}$ is complete,*

(iii) *$\tilde{\mathscr{G}}$ and $\mathscr{G}$ have the same causally connected sets,*

(iv) *if $\tilde{\mathscr{G}}'$ is a causality in X which satisfies* (i)–(iii), *then $\tilde{\mathscr{G}}'$ is stronger than $\tilde{\mathscr{G}}$.*

Proof: Every causally connected set in $\mathscr{G}$ is contained in a maximal causally connected set. Let $\mathscr{C}$ be a family of all maximal causally connected sets and let $\tilde{\mathscr{G}} = \mathscr{G} \cup \mathscr{C}$. If A is causally connected in $\mathscr{G}$, then A is causally connected in $\tilde{\mathscr{G}}$. Conversely, if A is causally connected in $\tilde{\mathscr{G}}$ and x, $\mathscr{G}$ are in A, then there is some path $\tilde{\Gamma}$ in $\tilde{\mathscr{G}}$ which connects x and y. If $\tilde{\Gamma} \notin \mathscr{G}$, then $\tilde{\Gamma}$ is causally connected in $\mathscr{G}$ and so there is $\Gamma \in \mathscr{G}$ which connects x and y, and so (iii) holds.

Let us show that $\tilde{\mathscr{G}}$ is complete. In fact, if A is causally connected in $\mathscr{G}$, then A is causally connected in $\tilde{\mathscr{G}}$ and therefore, there exists a maximal causally connected set $\tilde{\Gamma}$ containing A. But then $\tilde{\Gamma}$ is a path in $\tilde{\mathscr{G}}$, and so A is causal in $\tilde{\mathscr{G}}$. Finally, if $\tilde{\mathscr{G}}'$ is as in (iv) and $\tilde{\Gamma} \in \mathscr{C}$, then $\tilde{\Gamma}$ is causally connected, and so there is a path $\tilde{\Gamma}'$ in $\tilde{\mathscr{G}}'$ which contains $\tilde{\Gamma}$. But $\tilde{\Gamma}'$ is causally connected and $\tilde{\Gamma}$ is a maximal causally connected set, so it follows that $\tilde{\Gamma} = \tilde{\Gamma}'$. ■

The causality $\tilde{\mathscr{G}}$, uniquely defined by (i)–(iv), is said to be a *completion* of $\mathscr{G}$. Let us observe that if $\mathscr{G}$ is already complete, then $\mathscr{G}$ coincides with $\tilde{\mathscr{G}}$.

2a. Orthogonality spaces (see Greechie [4])

We review here the main facts and definitions related to a concept of an orthogonality space. An orthogonality space is a pair $(X, \perp)$, where $\perp$ is a symmetric irreflexive relation in X. If $x \perp y$, then x and y are said to be *orthogonal*, and x is said to be *orthogonal to A* if x is orthogonal to every point of A. If every point of A is orthogonal to B, then A is said to be *orthogonal to B*, and the set of all points orthogonal to A, which is the largest set orthogonal to A, is called the *orthogonal complement* of A and denoted by $A^\perp$. It follows from the definition that the mapping $A \to A^\perp$ of 2^X into itself has the following properties:

(i) $A \subset A^{\perp\perp}$,

(ii) $A \subset B \Rightarrow B^\perp \subset A^\perp$,

(iii) $A \cap A^\perp = \emptyset$,

(iv) $(\bigcup A_i)^\perp = \bigcap A_i^\perp$,

(v) $A^\perp = A^{\perp\perp\perp} = \ldots$,

(vi) $(A \cap B)^\perp \supset A^\perp \cup B^\perp$,

(vii) *if* $A_i = A_i^{\perp\perp}$, *then* $(\bigcap A_i)^\perp = (\bigcup A_i^\perp)^{\perp\perp}$.

PROPOSITION 2. *Every mapping $A \to A^{\perp}$ in 2^X which satisfies* (i)–(iv) *is generated by a uniquely defined orthogonality relation in X.*

Proof: The relation "$x \perp y$ if and only if $x \in \{y\}^{\perp}$" is symmetric by (i) and (ii), and irreflexive by (iii). It generates $A \to A^{\perp}$ owing to (iv). ■

A subset A of an orthogonality space $(X, \perp)$ is said to be *orthogonal* (or, said to be a *$\perp$-set*) if every two distinct points of A are orthogonal. Every orthogonal set is contained in a maximal orthogonal set. The family of all $S^{\perp\perp}$, where S is a $\perp$-set, is said to be *quasilogic* of X denoted by $\mathscr{L}$.

Let $(X, \perp)$ be an orthogonality space and let "$x \perp' y$" mean that $x \neq y$ and x is not orthogonal to y. Thus "$\perp'$" is also an orthogonality relation in X. The set A is said to be a *$\perp$-admissible path* if A is a $\perp'$-set. If for every two points x and y which can be connected by a $\perp$-admissible path and for every maximal $\perp$-set S there exists a $\perp$-admissible path intersecting S, then $(X, \perp)$ is said to be a *D-space*. If every maximal $\perp$-admissible path intersects every maximal $\perp$-set, then $(X, \perp)$ is called an *F-space*. In a D-space $A \in \mathscr{L}$ implies $A^{\perp} \in \mathscr{L}$, and $\mathscr{L}$, when ordered by set theoretic inclusion, is an orthomodular poset.

Let $(X, \mathscr{G})$ be a causality space. There exists a natural orthogonality relation in X: $x \perp y$ if and only if x and y are not causally related. We say that "$\perp$" is generated by $\mathscr{G}$.

Let "$\perp$" be an orthogonality relation in X and let $\mathscr{G}_{\perp}$ be the family of all $\perp$-admissible paths. Then $\mathscr{G}_{\perp}$ is a complete causality in X, and $\perp$ is generated by $\mathscr{G}_{\perp}$. Let us note that in a causality space, "$x \perp y$" means that $\{x, y\}$ is not a causally connected set. Thus causalities with the same causally connected sets have the same orthogonality relation.

Different complete causalities, with different control relations, may generate the same orthogonality relation.

Let $(X, \mathscr{G})$ be a causality space and let "$\perp$" be generated by $\mathscr{G}$. Then $(X, \perp)$ is a D-space if and only if for every maximal orthogonal set S, and for every pair of causally related points x and y, there is a $z \in S$ which is causally related to both x and y. If $\mathscr{G}$ is complete, the last condition means that if x and y can be connected by a path, then they can be connected by a path intersecting S, and $(X, \perp)$ is an F-space if and only if every path is contained in a path intersecting S (in other words, if every maximal set is a global Cauchy surface; see Penrose [6]).

Let us observe that in a causality space the operations $A \to A^{\perp}$ and $A \to D(A)$ commute:

$$D(A^{\perp}) = (D(A))^{\perp} = A^{\perp}.$$

Thus, $\mathscr{L} \subset \tilde{\mathscr{L}}$ and, in fact, we have $\mathscr{L} \subset \mathscr{L}_1 \subset \tilde{\mathscr{L}}$, where $\mathscr{L}_1$ is defined as the family of all those $A \subset X$ for which $A = A^{\perp\perp}$. It follows that for every two sets A and B in $\mathscr{L}_1$, $A \alpha B$ is equivalent to $A \subset B$ and so, if one confines oneself to the study of $\mathscr{L}$ or $\mathscr{L}_1$, then the orthogonality relation is the only relation one needs to concern (except the natural structure of 2^X). In a persistent causality space, which is an F-space, each path intersects every maximal orthogonal set and so, if S is an orthogonal set, then $D(S) = S^{\perp\perp}$.

3. Galilean causal logic

It is convenient to identify the Galilean space-time X with the tensor product $\boldsymbol{R} \times \boldsymbol{R}^3$, and to characterize every point x of X by its time coordinate x^0 and space coordinates $\boldsymbol{x}(x^1, x^2, x^3)$. The two distinct points x and y are orthogonal if and only if $x^0 = y^0$. Every hyperplane $S_{x^0} = \{y; y^0 = x^0\}$ is a maximal $\perp$-set, and every maximal $\perp$-set is of this form. Every section of the projection π^0: $x \to x^0$ (or rather its image) is a maximal $\perp$-admissible path, and every maximal $\perp$-admissible path is of this form. Thus, $(X, \perp)$ is an F-space, and the logic $\mathscr{L}$ of $(X, \perp)$ coincides with the family of all causal closures of orthogonal sets. If S is an orthogonal set, then $S \subset S_{x^0}$ for some x^0 and either $S = S_{x^0}$ and therefore $D(S) = X$, or $S \neq S_{x^0}$ and then $S^{\perp} = S_{x^0} - S$ and so $D(S) = S^{\perp\perp} = S$. The set $T_y = \{z; \boldsymbol{z} = \boldsymbol{y}\}$ is then a path not intersecting S. Therefore, $D(S) = S$, and so there is a natural isomorfism of $\mathscr{L}$ onto a disjoint union of Boolean logics $\mathscr{L}_{x^0} = 2^{S_{x^0}}$, S_{x^0}-s being identified with:

$$\mathscr{L} = \bigcup_{x^0} \mathscr{L}_{x^0}.$$

We note that there are no relations between non-trivial elements of different $\mathscr{L}_{x^0}$-s. Thus $\mathscr{L}$ is a complete lattice. It is reasonable to restrict ourselves to the Borel sets in X, the corresponding logic is denoted by $\mathscr{L}^B$.

The action of the Galilean group G on X is given by:

$$(R, \eta, \boldsymbol{v}, \boldsymbol{a}): (x_0, \boldsymbol{x}) \to (x^0 + \eta, R\boldsymbol{x} + \eta \boldsymbol{v} + \boldsymbol{a})$$

and if $g = (R, \eta, \boldsymbol{v}, \boldsymbol{a})$, $g' = (R', \eta', \boldsymbol{v}', \boldsymbol{a}')$ then

$$gg' = (RR', \eta + \eta', R\boldsymbol{v}' + \boldsymbol{v}, R\boldsymbol{a}' + \boldsymbol{a} + \eta' \boldsymbol{v}).$$

Since the map $x \to gx$ is an orthogonality preserving homeomorphism of X, it follows that G acts as a group of automorphisms of $\mathscr{L}$ and $\mathscr{L}^B$.

Let us now discuss in some detail representations of the Galilean logic. To a localizable Galilean quantum system there should correspond a representation E: $S \to E(S)$ of $\mathscr{L}^B$ by projections in the Hilbert space $\mathscr{H}$ of the system: if $\{S_n\}$ is a sequence of pairwise orthogonal elements of $\mathscr{L}^B$, then $E(S_n)$ are pairwise orthogonal and $E(\vee S_n) = \sum E(S_n)$.

It follows then, by the weak modularity of $\mathscr{L}^B$, that $S \alpha S'$ implies $E(S) \alpha E(S')$. Owing to the specific structure of $\mathscr{L}^B$, it is easy to characterize the most general form of a representation E. In fact, if E is a representation of $\mathscr{L}^B$ in $\mathscr{H}$, then E_{x^0} (the restriction of E to $\mathscr{L}_{x^0}$) is a Borel spectral measure on $\boldsymbol{R}^3$. Conversely, if $\{E_{x^0}\}_{x^0 \in R}$ is an arbitrary family of Borel spectral measures on $\boldsymbol{R}^3$ then E defined by: $E(S) := E_{\pi_0(S)}(\boldsymbol{\pi}(S))$ is a representation of $\mathscr{L}^B$, where $\boldsymbol{\pi}$ is a projection $\boldsymbol{\pi}$: $(x^0, \boldsymbol{x}) \to \boldsymbol{x}$. In general, different dynamics shall lead to inequivalent representations of $\mathscr{L}^B$. However, if the external field is not too singular, the translational symmetry $(x^0, \boldsymbol{x}) \to (x^0, \boldsymbol{x} + \boldsymbol{a})$, when restricted to each of $\mathscr{L}_{x^0}$-s, should be still unitarily implementable. If, in addition, the external field is time independent, and the representation E is irreducible, E is necessarily of the following form:

$$\mathscr{H} = L^2(\mathbf{R}^3, \mathscr{K}, d^3x),$$
$$\big(E_0(\boldsymbol{\pi}(S))f\big)(\mathbf{x}) = \chi_{\pi(s)}(x)f(\mathbf{x}),$$
$$E_{x^0}(\boldsymbol{\pi}(S)) = U(x^0)E_0(\boldsymbol{\pi}(s))U(x^0)^*,$$

where $\mathscr{K}$ is some Hilbert space and $x^0 \to U(x^0)$ is a continuous unitary representation of the time-translation group. The only restriction on a dynamics we get from the above assumptions is rather weak: if $\boldsymbol{q}(x^0)$ is the position operator at an instant x^0, then $\{\boldsymbol{q}(x^0)\}_{x^0\in \mathbf{R}}$ should be an irreducible set. Excluded are thus Hamiltonians H, which are functions of $q(x^0)$ only. An interesting and satisfactory restriction on the form of H can be obtained by restricting to those representations of $\mathscr{L}^B$ which admit a conserved current.

4. Covariant representations of the Galilean logic

By a *covariant representation* of $\mathscr{L}^B$ we mean a unitary representation $g \to U(g)$ of the Galilei group G and a representation E: $S \to E(S)$ of $\mathscr{L}^B$ by projections such that

$$U_g E(S) U_g^* = E(g\cdot S).$$

In general, U_g is a projective representation, with a multiplier, which commutes with $\{E(S)\}_{S\in L^B}$. We shall assume that the multiplier of U is a c-number function. Our goal is then to find a most general form of E and U. According to Bargmann [1], the most general form of a projective representation of G can be obtained from a projective representation of its covering group $\tilde{G}$ with a multiplier

$$U(g)U(g') = e^{i\alpha\xi(g,g')}U(gg'),$$

where

$$g = (A, \eta, \boldsymbol{v}, \boldsymbol{a}),$$
$$g' = (A', \eta', \boldsymbol{v}', \boldsymbol{a}), \tag{$*$}$$
$$\xi(g, g') = \eta\left(\frac{v'^2}{2} + \boldsymbol{v}\cdot R_A\boldsymbol{v}'\right) - (\boldsymbol{a}\cdot R_A\boldsymbol{v}'),$$

and $A \to R_A$ is a $2\to 1$ homomorphism of SU(2) onto SO(3). We shall consider only the case of $\alpha \neq 0$. It can easily be seen that, owing to the relations ($*$), in order to have

$$U(g)E(S)U(g)^* = E(g\cdot S) \quad \forall S\in\mathscr{L},$$

it is sufficient to assume that for all $S\in\mathscr{L}_0$

$$U(A, 0, \boldsymbol{v}, \boldsymbol{a})E_0(S)U(A, 0, \boldsymbol{v}, \boldsymbol{a})^* = E_0(R_A S + \boldsymbol{a}) \tag{$**$}$$

and then to define $E_{x^0}(S)$ by

$$E_{x^0}(S) = U(x_0)E_0(\boldsymbol{\pi}(S))U(x^0)^*.$$

In order to simplify the discussion, let us replace ($*$) and ($**$) by their infinitesimal counterparts, i.e. a representation of an extension of the Lie algebra of G:

(1) $[J_i, J_j] = i\varepsilon_{ijk}J_k$, (5) $[H, K_i] = -iP_i$, (9) $[P_i, P_j] = 0$,
(2) $[J_i, H] = 0$, (6) $[H, P_i] = 0$,
(3) $[J_i, K_j] = i\varepsilon_{ijk}K_k$, (7) $[K_i, P_j] = i\alpha\delta_{ij}$,
(4) $[J_i, P_j] = i\varepsilon_{ijk}P_k$, (8) $[K_i, K_j] = 0$,

and the commutation relations

(10) $[J_i, Q_j] = i\varepsilon_{ijk}Q_k$, (12) $[P_i, Q_j] = -i\delta_{ij}$,
(11) $[K_i, Q_j] = 0$, (13) $[Q_i, Q_j] = 0$,

where

$$\boldsymbol{Q} = \int \boldsymbol{q}\, dE_0(\boldsymbol{q}).$$

Since $\boldsymbol{P}$ and $\boldsymbol{Q}$ satisfy the canonical commutation relations (9), (12), (13), it follows by the theorem of von Neumann that $\mathscr{H}$ can be identified with $L^2(\mathbf{R}^3, \mathscr{K}, d^3x)$ (the Hilbert space of square integrable functions on $\mathbf{R}^3$ with values in a Hilbert space $\mathscr{K}$), and

$$(P_i f)(\boldsymbol{p}) = p_i f(\boldsymbol{p}),$$
$$(Q_i f)(\boldsymbol{p}) = i\frac{\partial f}{\partial p_i}.$$

Let $\boldsymbol{M} = \boldsymbol{Q}\times\boldsymbol{P}$; then $\boldsymbol{S} = \boldsymbol{J}-\boldsymbol{M}$ commutes with $\boldsymbol{Q}$ and $\boldsymbol{P}$ and so acts in $\mathscr{K}$ only. Moreover, from (1), (4), (10) we get $[S_i, S_j] = i\varepsilon_{ijk}S_k$. Now, let $\boldsymbol{k} = \boldsymbol{K}-\alpha\boldsymbol{Q}$; then, by (7), (11), (12), it follows that $\boldsymbol{k}$ acts on $\mathscr{K}$ only and that $[k_i, k_j] = 0$, $[s_i, k_j] = i\varepsilon_{ijk}k_k$.

Finally, we must satisfy (2), (5), and (6). By (6), H is a function $h(\boldsymbol{p})$ of the variables $\boldsymbol{p}$, and with $v(\boldsymbol{p}) = h(\boldsymbol{p})-p^2/2m$ and $T = e^{\frac{i}{\alpha}pk}$, we easily find that $(T^{-1}vT)\boldsymbol{p}$ commutes with $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{S}$. In particular, $(T^{-1}vT)(\boldsymbol{p}) \equiv v_0$, where v_0 acts on $\mathscr{K}$ only. The considerations above can be summarized as follows:

PROPOSITION 3. *Let $\mathscr{K}$ be a Hilbert space and let $W(A)$ and $W(\boldsymbol{v})$ be a unitary representation of the Euclidean group of $\mathbf{R}^3$ in $\mathscr{K}$. Let v_0 be a self-adjoint operator in $\mathscr{K}$, which commutes with the rotations $W(A)$, and let $W(x^0) = \exp(iv_0x^0)$. Let $\mathscr{H} = L^2(\mathbf{R}^3, \mathscr{K}, d^3x)$ and let $V(A, \eta, \boldsymbol{v}, \boldsymbol{a})$ be the canonical representation of $\tilde{G}$ in $\mathscr{H}$, with the multiplier $e^{i\xi(g,g')}$ given by ($*$), i.e.*

$$\big(V(A, \eta, \boldsymbol{v}, \boldsymbol{a})f\big)(\boldsymbol{p}) = \exp\left(i\left(\frac{p^2}{2\alpha} - \frac{\alpha v^2}{2}\right)\eta\right)\exp\big(-i(\boldsymbol{p}-\alpha\boldsymbol{v})\boldsymbol{a}\big)f\big(R_A^{-1}(\boldsymbol{p}-\alpha\boldsymbol{v})\big).$$

Let $V(\boldsymbol{p})$ be defined by

$$\big(V(\boldsymbol{p})f\big)(\boldsymbol{p}') = f(\boldsymbol{p}'-\boldsymbol{p})$$

and let E_0 be the spectral measure of the operators $\boldsymbol{Q} = i\partial/\partial p$, i.e.

$$V(\boldsymbol{p}) = \int e^{ipx}dE_0(x).$$

Finally, let

$$T = \int W(p/\alpha)dE(p),$$

where $E(p)$ is the canonical spectral measure in $\mathscr{H}$ (i.e. multiplication by characteristic functions). Then

$$U(A) = V(A)W(A),$$
$$U(\eta) = V(\eta)TW(\eta)T^{-1},$$
$$U(v) = V(v)W(v),$$
$$U(a) = V(a),$$

and

$$E_\eta(S) = U(\eta)E_0(\pi(S))U(\eta)^*$$

is a covariant representation of the logic $\mathscr{L}$. Every covariant representations of $\mathscr{L}$ is of this form (assuming $\alpha \neq 0$). ∎

From the physical point of view, a covariant representation of $\mathscr{L}$ of the above form describes a localization of a fixed point of a rigid body. The operator T is connected with a transformation to the center of mass coordinate system. An interesting example is obtained if one takes W to be an irreducible representation of the Euclidean group corresponding to spin zero and $\boldsymbol{k}^2 = k_0^2 \neq 0$:

$$\mathscr{K} = L^2(S_{k_0}, d\Omega(\boldsymbol{k})),$$
$$(k_i f)(\boldsymbol{k}) = k_i f(\boldsymbol{k}),$$
$$\boldsymbol{S} = -i\left(\boldsymbol{k} \times \frac{\partial}{\partial \boldsymbol{k}}\right),$$

where $S_{k_0} = \{\boldsymbol{k}; \boldsymbol{k}^2 = k_0^2\}$. Every scalar operator v_0 in $\mathscr{K}$ is a function of S^2 and in the simplest nontrivial case we can take

$$v_0 = \beta\frac{S^2}{k_0^2}, \quad \beta \neq 0.$$

Then, the Hamiltonian H is given by

$$H = P^2\left(\frac{1}{2\alpha}+\frac{\beta}{\alpha^2}\right)+\frac{\beta}{k_0^2}S^2-\frac{2\beta}{\alpha k_0^2}[\boldsymbol{P}\cdot[\boldsymbol{s}\times\boldsymbol{k}]+H.C.]-\frac{\beta}{\alpha^2 k_0^2}(\boldsymbol{P}\cdot\boldsymbol{k}).$$

If $\beta \neq -\alpha/2$, this Hamiltonian describes the evolution of a point of mass $m_1 = \alpha^2/(\alpha+2\beta)$ rigidly connected with a second point of mass $m_2 = \alpha - m_1$, k_0^2/m_2^2 being the distance between the two masses. If $\beta = -\alpha/2$, we get a system of two infinite masses (of opposite signs) the total mass of the system being finite.

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