Vol.25 (1987) *REPORTS ON MATHEMATICAL PHYSICS* No. 1

GRADED LIE-CARTAN PAIRS I

ARKADIUSZ JADCZYK *

Institute of Theoretical Physics, University of Wrocław

and

DANIEL KASTLER **

Physics Department, University of Aix-Marseille II (Luminy)

and

Centre de Physique Théorique***

(*Received January 12, 1987*)

The Lie–Cartan pairs proposed in [1] as an algebraic frame for the classical operators of differential geometry are generalized to the $Z/2$-graded case (graded Lie–Cartan pairs of a graded Lie algebra and a graded commutative algebra). The generalized case is reduced to the Abelian case by tensoring with arbitrary graded commutative algebras.

The Lie–Cartan pairs (L, A) of a Lie algebra and a commutative algebra were introduced in [1] as a purely algebraic frame for describing the interrelation between the following classical geometric items: the covariant exterior derivative δ_ϱ attached to a connection ϱ (resp. the exterior derivative δ), the Lie derivative $\theta(\xi)$, and the inner product $i(\xi)$. The (associative) algebra A in [1] was assumed commutative because of the need, for a derivation ξ of A, to yield again a derivation $a\xi$ through composition with multiplication from the left by any element a of A. However this fact holds more generally for A *graded commutative*, provided we take derivations in the graded commutative sense (thus yielding a

* Postal address: Institute of Theoretical Physics, University of Wrocław, Cybulskiego 36, 50-205 Wrocław.

** Postal address: CPT–CNRS Luminy Case 907-F-13288 Marseille Cedex 9.

*** Laboratoire Propore du Centre National de la Recherche Scientifique.

Lie super algebra structure[1]). With this choice the theory in [1] can be generalized (with identical results up to $\mathbf{Z}/2$-grading to "*graded Lie–Cartan pairs*" (L, A) of a Lie super algebra L and a graded commutative algebra A, with definition axioms being natural modifications of the definition axioms in [1]: one needs only add, wherever necessary, the characteristic signs $(-1)^{ij}$ concomitant with permutation of factors with respective grades i and j. Now, instead of adapting the proofs in [1] to the $\mathbf{Z}/2$-graded case – which would lead to cumbersome calculations, one can, more interestingly, reduce the generalized $\mathbf{Z}/2$-graded results to the previous (trivially graded) case by means of the following device (possibly of independent interest): given a graded Lie–Cartan pair (L, A) and an arbitrary graded commutative algebra U, the pair (L_U, A_U) obtained from the skew products $L_U = U \otimes L$, $A_U = U \otimes A$, turns out to be again naturally a graded Lie–Cartan pair, and this in such a way that:

(i) V-connections of (L, A) naturally define corresponding V_U-connections of (L_U, A_U) for $V_U = U \otimes V$;

(ii) graded-alternate A-linear V-valued n-forms λ on L naturally yield corresponding A_U-linear V_U-valued forms λ_U on L_U, the operators δ_ϱ, $\theta_\varrho(\xi)$ and $i(\xi)$ essentially commuting with the map $\lambda \to \lambda_U$.

Moreover, since knowledge of all restrictions λ_U to zero grade elements characterizes λ, one has automatic transfer of results from the trivially graded to the $\mathbf{Z}/2$-graded case.

Our paper is organized as follows: Section 1 defines the graded Lie–Cartan pairs, discusses adjuction of a unit, and describes the injective and degenerate special cases. Section 2 defines the classical operators δ_ϱ, δ_0, $\theta_\varrho(\xi)$, $i(\xi)$ and provides the main Theorem 2.3. We formulate the definitions in terms of the graded antisymmetrizer A_n, so as to obtain operators a priori respecting the graded-alternate property – with the bonus of thus motivating the complicated signs arising in the explicit formulae (2.2) specifying these operators. Section 3 describes the passage of pairs (L, A) to pairs (L_U, A_U), the assignment of V_U-connections of (L_U, A_U), to V-connections of (L, A), and the extension $\lambda \to \lambda_U$ of graded alternate forms, with the ensuing commutation theorem (independent of Sections 1 and 2 in which it is used for the proof of Theorem 2.3). Section 4 discusses derivation properties of δ_ϱ, $\theta_\varrho(\xi)$, and $i(\xi)$. We gathered in Appendix A necessary results on graded vector spaces and algebras. Appendix B describes "graded symmetrization" in terms of more general "twisted symmetrization".

In our paper we do not discuss the concepts of graded manifolds or supermanifolds since we do not need to use them. However, for the benefit of the reader, we have given some of the recent references (see Refs. [2–8]) on the subject. In particular, Ref. [6] relates to the tensorization method which we discuss in Section 3.

---

[1] Since the only gradings appearing in this paper are $\mathbf{Z}/2$-gradings, we shall indifferently use the

1. Graded Lie–Cartan pairs

1.1. DEFINITION. A *real* (*complex*) *graded Lie–Cartan pair*[2] is a couple (L, A) of a graded Lie algebra L and a unital graded commutative algebra A, both real (complex), endowed with graded bilinear products:

$$(\xi, a) \in L \times A \to \xi a \in A, \quad \text{with } L^i A^j \subset A^{i+j}, \quad i, j \in \mathbf{Z}/2 \tag{1.1}$$

and

$$(a, \xi) \in A \times L \to a\xi \in L, \quad \text{with } A^i L^j \subset L^{i+j}, \tag{1.2}$$

with the following properties:

(i) the product (1.1) defines a homomorphism[3] $L \to \operatorname{Der} A$ of graded Lie algebras, i.e. one has

$$\xi(a, b) = (\xi a) b + (-1)^{\partial a \partial \xi} a(\xi b), \quad \xi \in L^{\cdot}, a \in A^{\cdot}, b \in A, \tag{1.3}$$

$$[\xi, \eta] a = \xi(\eta a) - (-1)^{\partial \xi \partial \eta} \eta(\xi a), \quad a \in A, \xi \in L^{\cdot}, \eta \in L, \tag{1.4}$$

(ii) the product (1.2) makes L a unital left A-module[4]

$$a(b\xi) = (ab)\xi \ (\text{denoted } ab\xi), \quad a, b \in A, \xi \in L, \tag{1.5}$$

$$1\xi = \xi, \quad \xi \in L, \tag{1.6}$$

(iii) we have in addition the properties[5]

$$(a\xi) b = a(\xi b) \ (\text{denoted } a\xi b), \quad a, b \in A, \xi \in L, \tag{1.7}$$

$$[\xi, a\eta] = (-1)^{\partial a \partial \xi} a[\xi, \eta] + (\xi a)\eta, \quad a \in A^{\cdot}, \xi \in L^{\cdot}, \eta \in L. \tag{1.8}$$

With (L, A) a graded Lie–Cartan pair, and V a linear graded left A-module,[6] a V-connection is a zero-grade assignment, to each $\xi \in L$ of a linear operator $\varrho(\xi)$[7]

---

[2] We currently use the term graded to mean *$\mathbf{Z}/2$-graded* (we recall that we denote by $E^{\cdot}$ the set $E^0 \cup E^1$ of homogeneous elements of the graded vector space $E = E^0 + E^1$). For a general information on graded structures see Appendix.

[3] In fact a zero-grade homomorphism according to the second line (1.1).

[4] In fact a graded left A-module according to the second line (1.2).

[5] The axiom (1.8) together with the commutation axiom of graded Lie algebras implies that one has $[a\xi, \eta] = a[\xi, \eta] - (-1)^{\partial\eta(\partial a + \partial\xi)} (\eta a)\xi$.

[6] I.e. a graded real (complex) vector space carrying a linear representation $a \in A \to [X \in V \to aX \in V]$ with $A^p V^k \subset A^{p+k}$, $p, k \in \mathbf{Z}/2$.

[7] I.e. $\mathbf{R}$-linear ($\mathbf{C}$-linear). Note that since A is assumed unital, the second line in (1.9) could be

of V behaving as a derivation for the left A-module structure of V:

$$\begin{aligned}\varrho(\xi)X &\subset V^{\partial\xi\,\partial X},\\ \varrho(\xi)(\alpha X+\beta Y) &= \alpha\varrho(\xi)X+\beta\varrho(\xi)Y,\\ \varrho(\xi)(aX) &= (-1)^{\partial\xi\,\partial a}\varrho(\xi)X+(\xi a)X,\end{aligned}\tag{1.9}$$

where

$$\xi\in L^{\cdot},\ X\in V^{\cdot},\ Y\in V,\ a\in A^{\cdot},\ \alpha,\ \beta\in \boldsymbol{R}(\boldsymbol{C}),$$

and moreover linear in the sense

$$\varrho(\alpha\xi+\beta\xi)=\alpha\varrho(\xi)+\beta\varrho(\xi),\qquad \xi,\ \eta=L,\ \alpha,\ \beta\in\boldsymbol{R}(\in\boldsymbol{C}).\tag{1.10}$$

The V-connection ϱ is called *local* whenever

$$\varrho(a\xi)=a\varrho(\xi),\qquad a\in A,\ \xi\in L\tag{1.11}$$

and *flat* whenever

$$\varrho([\xi,\eta])=[\varrho(\xi),\ \varrho(\eta)],\tag{1.12}$$

where [,] in the r.h.s. denotes a graded commutator.

The curvature of the V-connection ϱ is the assignment to $(\xi,\eta)\in L\times L$ of the map: $V\to V$ given by

$$\Omega(\xi,\eta)=[\varrho(\xi),\ \varrho(\eta)]-\varrho([\xi,\eta]),\tag{1.13}$$

where the first bracket on the r.h.s. denotes a graded commutator.

1.2. LEMMA. *Given a V-connection ϱ of a graded Lie–Cartan pair (L, A), the value $\Omega(\xi,\eta)$ of the curvature for $\xi,\ \eta\in L^{\cdot}$ behaves as follows w.r.t. the left A-module structure of V: one has*[8]

$$\Omega(\xi,\eta)(aX)=(-1)^{(\partial\xi+\partial\eta)}a\{\Omega(\xi,\eta)X\},\qquad a\in A^{\cdot},\ X\in V.\tag{1.14}$$

Proof: We have, writing $\partial a=i,\ \partial\eta=j,\ \partial\xi=k$

$$\begin{aligned}\eta(\xi)\{\varrho(\eta)(aX)\} &= \varrho(\xi)\{(-1)^{ij}a\varrho(\eta)X+(\eta a)X\}\\ &= (-1)^{i(k+j)}a\varrho(\xi)\varrho(\eta)X+(-1)^{ij}(\xi a)\varrho(\eta)X+\\ &\quad+(-1)^{k(i+j)}(\eta a)\varrho(\xi)X+\xi(\eta a)X,\end{aligned}\tag{1.15}$$

hence

$$\begin{aligned}[\varrho(\xi),\ \varrho(\eta)](aX) &= (-1)^{i(k+j)}a\{\varrho(\xi)\varrho(\eta)-(-1)^{jk}\varrho(\eta)\varrho(\xi)\}X+\\ &\quad+(-1)^{k(i+j)}(\eta a)\varrho(\xi)-\\ &\quad-(-1)^{k(i+j)+jk}(\xi a)\varrho(\eta)X+[\xi(\eta a)-(-1)^{jk}\eta(\xi a)]X\\ &= (-1)^{j(k+j)}a[\varrho(\xi),\ \varrho(\eta)]X+([\xi,\eta]a)\mathrm{X}.\end{aligned}\tag{1.16}$$

[8] In other terms, $\Omega(\xi,\eta)$ is an endomorphism of grade $\partial\xi+\partial\eta$ of the left A-module V.

On the other hand

$$\varrho([\xi,\eta])(aX)=(-1)^{i(k+j)}a\varrho([\xi,\eta])X+([\xi,\eta]a)X,\tag{1.17}$$

whence (1.14) by difference.

1.3. *Remark*. The first example of a real (complex) Lie–Cartan pair (L, A) is obtained as follows: take for A a graded-commutative algebra, with $L=\mathrm{Der}\,A$, the set of graded derivations of A, [,] the graded commutator. We recall that $\mathrm{Der}\,A=(\mathrm{Der}\,A)^0\oplus(\mathrm{Der}\,A)^1$ with $(\mathrm{Der}\,A)^0$, resp. $(\mathrm{Der}\,A)^1$ the even, resp. odd endomorphisms ξ of A as a graded vector space such that

$$\xi(ab)=(Xa)b+a(\xi b),\qquad a,\ b\in A,\tag{1.18}$$

resp.

$$\xi(ab)=(Xa)b+(-1)^{\partial a}a(\xi b),\qquad a\in A^{\cdot},\ b\in A.\tag{1.19}$$

For these facts, we refer to Appendix A.

1.4. DEFINITION. A *subpair* of a graded Lie–Cartan pair (L, A) is a couple $(L^{\cdot}, A^{\cdot})$ of a subsuper Lie algebra $L^{\cdot}$ of L and a graded unital subalgebra $A^{\cdot}$ of A such that one has $\xi a\in A^{\cdot}$ and $a\xi\in L^{\cdot}$ for all $\xi\in L^{\cdot}$ and $a\in A^{\cdot}$. The subpair $(L^{\cdot}, A^{\cdot})$ is itself a graded Lie–Cartan pair for the products (1.1) and (1.2).

Given a real (complex) graded Lie–Cartan pair (L, A) the subpair $(L, \boldsymbol{R})$ (resp. $(L, \boldsymbol{C})$) is called the *depletion of* (L, A).[9]

1.5. DEFINITION. We define, for a graded Lie–Cartan pair (L, A)

$$L^{\perp}=\{\xi\in L\colon\ \xi a=0\ \text{for all}\ a\in A\},\tag{1.20}$$

$$A^{\perp}=\{a\in A\colon\ \xi a=0\ \text{for all}\ \xi\in L\}.\tag{1.21}$$

The pair (L, A) is called *injective* whenewer $L^{\perp}=\{0\}$, and *degenerate* whenever $L^{\perp}=L$ (or equivalently $A^{\perp}=\{0\}$).

1.6. LEMMA. *The Definition 1.5 implies*

(i) *$L^{\perp}$ is a graded ideal of the graded Lie algebra L and a submodule of the left A-module L.*

(ii) *$A^{\perp}$ is a graded unital subalgebra of A.*

(iii) *L is a graded Lie algebra over $A^{\perp}$ (i.e. the bracket [,] of L is graded, $A^{\perp}$-bilinear).*

(iv) *The products (1.1) and (1.2) are also graded $A^{\perp}$-bilinear.*

(v) *$(L/L^{\perp}, A)$ is an injective graded Lie–Cartan pair with the definitions*

[8] Note that, for the depletion of (L, A), the product (1.1) is trivial: $\xi\alpha=0$ for all $\xi\in L$, $\alpha\in\boldsymbol{R}$ (resp. $\alpha\in\boldsymbol{C}$), owing to the fact that derivations of a unital algebra vanish on the unit.

$$\begin{aligned}[\bar{\xi}, \bar{\eta}] &= [\xi, \eta]^{-},\\ \bar{\xi}a &= \xi a, \\ a\bar{\xi} &= \overline{a\xi},\end{aligned} \tag{1.22}$$

where $\bar{\xi}$ denotes the class of $\xi \in L$ modulo $L^{\perp}$.

(vi) *The injective graded Lie–Cartan pairs are pairs of the type (L, A) with A a graded-commutative unital algebra, and L a Lie subsuperalgebra of* Der A.

(vii) *The degenerate-graded Lie–Cartan pairs are obtained from couples L, A of a Lie superalgebra L over A with $\xi a = 0$ for $\xi \in L$, $a \in A$.*

Proof: (i) $L^{\perp}$ is a graded ideal of L as the kernel of the homomorphism $\xi \to (a \to \xi a)$ from L to Der A. It is a submodule of the A-module L since it is obviously linear and, for $\xi \in L^{\perp}$, $(a\xi)b = a(\xi b) = 0$ for all $a, b \in A$.

(ii) $A^{\perp}$ is obviously linear. For $a = a^0 + a^1 \in A^{\perp}$, $\xi \in L^{\cdot}$ one has, $\xi a^0 = -\xi a^1 \in A^0 \cap A^1 = \{0\}$ thus $A^{\perp}$ is graded. And, for a, b in $A^{\perp}$, one has $\xi(ab) = (\xi a)b + (-1)^{\partial a \partial \xi} a(\xi b) = 0$ for all $\xi \in L^{\cdot}$. $A^{\perp}$ is unital, since $\mathbf{R} \subset A^{\perp} (\mathbf{C} \subset A^{\perp})$ (cf. footnote 9).

(iii) For ξ, $\eta = L^{\cdot}$ and $a \in A^{\perp}$ one has by (1.8)

$$\begin{aligned}[\xi, a\eta] &= (-1)^{\partial a \partial \xi} a[\xi, \eta],\\ [a\xi, \eta] &= a[\xi, \eta].\end{aligned} \tag{1.23}$$

(iv) One has, for $a \in A^{\perp}$, $b \in A^{\cdot}$, $\xi \in L^{\cdot}$, by (1.3) and (1.5)

$$\begin{aligned}(a\xi)b &= a(\xi b),\\ \xi(ab) &= (-1)^{\partial a \partial \xi} a\xi b\end{aligned} \tag{1.24}$$

and (in fact for all a, $b \in A^{\cdot}$)

$$\begin{aligned}a(b\xi) &= (-1)^{\partial a \partial b} ba\xi,\\ (ba)\xi &= b(a\xi).\end{aligned} \tag{1.25}$$

(v) The first line in (1.22) defines the bracket of the quotient superalgebra $L/L^{\perp}$. The two other lines are coherent definitions, since, for $\eta \in L^{\perp}$

$$(\xi + \eta)a = \xi a \quad \text{and} \quad \overline{a(\xi + \eta)} = \overline{a\xi},$$

since a, $\eta \in L^{\perp}$. Moreover one has, for a, $b \in A$, $\xi \in L$

$$(a\bar{\xi})b = \overline{a\xi}\, b = (a\xi)b = a(\xi b) = a(\bar{\xi} b) \tag{1.26}$$

and for $a \in A$, η, $\xi \in L$

$$\begin{aligned}[\bar{\xi}, a\bar{\eta}] = [\bar{\xi}, \overline{a\eta}] = [\xi, a\eta]^{-} &= (-1)^{\partial a \partial \xi} \overline{a[\xi, \eta]} + \overline{(\xi a)\eta}\\ &= (-1)^{\partial a \partial \xi} a \overline{[\xi, \eta]} + (\xi a)\bar{\eta} = (-1)^{\partial a \partial \xi} a [\bar{\xi}, \bar{\eta}] + (\bar{\xi} a)\bar{\eta}.\end{aligned} \tag{1.27}$$

(vi) Is obvious.

(vii) Is obvious from (iii). ∎

1.7. *Remark*. With (L, A) a graded Lie–Cartan pair and Q a subalgebra of $A^{\perp}$, Lemma 1.6 shows that the elements of Q behave like scalars. We say in such case that (L, A) is a *graded Lie–Cartan pair over* Q.

Remark. Most of the concepts and results of this paper admit an obvious generalization from the category of graded Lie algebras and graded vector spaces to the category of graded Lie algebras *over* Q and graded Q-modules (Q fixed). This kind of generalization is useful in applications in which anticommuting "parameters" are being used. Q is then usually taken as a Grassmann algebra with "sufficiently many" (possibly with an infinite number of) generators.

The next lemma shows that our assumptions, in Definition 1.1, that A is unital and L a unital A-module, do not in fact restrict generality.

1.8. LEMMA. *Define a non-unital graded Lie–Cartan pair as a pair (L, A) satisfying all the axioms in Definition 1.1 except the unital requirement of the algebra A and the left A-module L. Then $(L, \tilde{A})$ is a graded Lie–Cartan pair if one defines*:

(i) *$\tilde{A}$ as the algebra obtained from A by adding a unit* $\mathbf{1}$,

(ii) *the products* (1.1) *and* (1.2) *of $\xi \in L$ and $\alpha\mathbf{1} + a \in \tilde{A}$ as*

$$\xi(\alpha\mathbf{1} + a) = \xi a, \tag{1.28}$$

$$(\alpha\mathbf{1} + a)\xi = \alpha\xi + a\xi. \tag{1.29}$$

Moreover, with V a linear graded left A-module, the convention

$$(\alpha\mathbf{1} + a)X = \alpha X + aX, \qquad \alpha\mathbf{1} + a \in \tilde{A}, \qquad X \in V \tag{1.30}$$

makes V a unital graded left $\tilde{A}$-module, say $\tilde{V}$, each V-connection ϱ of (L, A) yielding a $\tilde{V}$-connection of $(L, \tilde{A})$.

Proof: Identical to that of Lemma 6 in [1]. ∎

2. The classical operators attached to a graded Lie–Cartan pair

In this section we present an abstract (purely algebraic) graded version of the classical operators, δ_{ϱ}, δ_0, $\theta_{\varrho}(\xi)$ and $i(\xi)$ of differential geometry.

2.1. DEFINITION. We fix a unital graded Lie–Cartan pair (L, A) and a unital graded left A-module V and denote by $\Lambda^n(L, V)$ the graded vector space of graded alternate V-valued n-linear forms on L, the range of the graded alternator A_n acting in $\mathscr{L}^n(L, V)$ (cf. Appendix B).

With $\xi,\ \eta \in L^{\cdot},\ a \in A^{\cdot}$ and ϱ a V-connection of (L, A) of curvature Ω, the operators $\delta_0,\ \varrho\wedge,\ \delta_\varrho,\ \theta_0(\xi),\ \varrho(\xi),\ \theta_\varrho(\xi),\ \Omega(\xi, \eta),\ i(\xi),\ a^{\cdot},\ \delta a\wedge$ and $\Omega\wedge$ are then defined as follows:[10] for $\lambda \in \Lambda^n(L, V)^{\cdot}$ and $\xi_i \in L^{\cdot},\ i = 1, \ldots, n+2$, we set

$$\delta_0\lambda = -\frac{n(n+1)}{2} A_{n+1}\lambda^\delta \quad \text{with } \lambda^\delta(\xi_1, \ldots, \xi_{n+1}) = \lambda([\xi_1, \xi_2], \xi_3, \ldots, \xi_{n+1}), \tag{2.1[11]}$$

$$\varrho\wedge\lambda = (n+1) A_{n+1}\lambda^\varrho \quad \text{with } \lambda^\varrho(\xi_1, \ldots, \xi_{n+1}) = (-1)^{\partial\lambda\partial\xi_1}\varrho(\xi_1)\lambda(\xi_2, \ldots, \xi_{n+1}), \tag{2.2}$$

$$\delta_\varrho = \delta_0 + \varrho\wedge, \tag{2.3}$$

$$\theta_0(\xi)\lambda = -nA_n\lambda^\xi \quad \text{with } \lambda^\xi(\xi_1, \ldots, \xi_n) = (-1)^{\partial\lambda\partial\xi}\lambda([\xi, \xi_1], \xi_2, \ldots, \xi_n), \tag{2.4}$$

$$(\varrho(\xi)\lambda)(\xi_1, \ldots, \xi_n) = \varrho(\xi)(\lambda(\xi_1, \ldots, \xi_n)), \quad (\Omega(\xi, \eta)\lambda)(\xi_1, \ldots, \xi_n) = \Omega(\xi, \eta)(\lambda(\xi_1, \ldots, \xi_n)), \tag{2.5}$$

$$\theta_\varrho(\xi) = \theta_0(\xi) + \varrho(\xi), \quad \{i(\xi)\lambda\}(\xi_1, \ldots, \xi_{n-1}) = (-1)^{\partial\lambda\partial\xi}\lambda(\xi, \xi_1, \ldots, \xi_{n-1}), \tag{2.6}$$

$$i(\xi)\lambda = 0, \quad \lambda \in \Lambda^0(L, V), \tag{2.7}$$

$$(a\cdot\lambda)(\xi_1, \ldots, \xi_n) = a\{\lambda(\xi_1, \ldots, \xi_n)\}, \tag{2.8}$$

$$\delta a\wedge\lambda = (n+1) A_{n+1}\lambda^a \quad \text{with } \lambda^a(\xi_1, \ldots, \xi_{n+1}) = (-1)^{\partial\xi_1(\partial a + \partial\lambda)}(\xi_1 a)\lambda(\xi_2, \ldots, \xi_{n+1}), \tag{2.9}$$

$$\Omega\wedge\lambda = \frac{(n+1)(n+2)}{2} A_{n+2}\lambda^\Omega \quad \text{with } \lambda^\Omega(\xi_1, \ldots, \xi_{n+2}) = (-1)^{\partial\lambda(\partial\xi_1 + \partial\xi_2)}\Omega(\xi_1, \xi_2)\lambda(\xi_3, \ldots, \xi_{n+2}). \tag{2.10}$$

2.2. PROPOSITION. *Let (L, A), V, ϱ, ξ and a be as in Definition 2.1. Then:*

[10] We recall that $E^{\cdot}$ denotes the set of homogeneous elements in the graded vector space E. Note that all Definitions 2.1 and 2.10 can be formulated for $\lambda \in \mathscr{L}^n(L, V)$.

[11] Implying the vanishing of δ_0 on $\Lambda^0(L, V)$. Similarly for $\theta_0(\xi)$.

(i) *the operators* (2.1) *through* (2.10) *are homomorphisms of the graded vector space $\Lambda^*(L, V)$: specifically, one has, for $\lambda \in \Lambda^n(L, V)$ (in fact, for $\lambda_0,\ \varrho\wedge,\ \delta_\varrho,\ \theta_0$, $\delta a\wedge$ and $\Omega\wedge$, for $\lambda \in \mathscr{L}^n(L, V)$)*[12]:

$$\begin{gathered} \delta_0\lambda,\ \varrho\wedge\lambda,\ \delta_\varrho\lambda \in \Lambda^{n+1}(L, V)^{\partial\lambda}, \\ \theta_0(\xi)\lambda,\ \varrho(\xi)\lambda,\ \theta_\varrho(\xi)\lambda \in \Lambda^n(L, V)^{\partial\lambda+\partial\xi}, \\ i(\xi)\lambda \in \Lambda^{n-1}(L, V)^{\partial\lambda+\partial\xi}, \\ a\cdot\lambda \in \Lambda^n(L, V)^{\partial\lambda+\partial a}, \\ \delta a\wedge\lambda \in \Lambda^{n+1}(L, V)^{\partial\lambda+\partial a}, \\ \Omega\wedge\lambda \in \Lambda^{n+2}(L, V)^{\partial\lambda}, \\ \Omega(\xi, \eta)\lambda \in \Lambda^n(L, V)^{\partial\lambda+\partial\xi+\partial\eta}, \end{gathered} \tag{2.11}$$

(ii) *one has for $\delta_0,\ \varrho\wedge,\ \theta_0(\xi),\ \delta a\wedge,\ \Omega\wedge$ the following explicit formulae: for $\xi_i \in L^{\cdot},\ i = 1, \ldots, n+2$ ($\hat{}$ indicating omission of the corresponding argument):*

$$(\delta_0\lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{1\leqslant i<j\leqslant n+1} (-1)^{\alpha_{ij}}\lambda([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{n+1}) \quad \textit{with } \alpha_{ij} = i+j+(\partial\xi_i+\partial\xi_j)\sum_{k=1}^{i-1}\partial\xi_k + \partial\xi_j\sum_{k=i+1}^{j-1}\partial\xi_k, \tag{2.12}$$

$$(\varrho\wedge\lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^{n+1}(-1)^{\beta_i}\varrho(\xi_i)\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}) \quad \textit{with } \beta_i = 1+i+\partial\xi_i\Big(\partial\lambda+\sum_{k=1}^{i-1}\partial\xi_k\Big), \tag{2.13}$$

$$\{\theta_0(\xi)\lambda\}(\xi_1, \ldots, \xi_n) = -\sum_{i=1}^{n}(-1)^{\gamma_i}\lambda(\xi_1, \ldots, \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, \ldots, \xi_n) \quad \textit{with } \gamma_i = \partial\xi\Big(\partial\lambda+\sum_{k=1}^{i-1}\partial\xi_k\Big), \tag{2.14}$$

[12] The relevant grading of $\Lambda^*(L, V)$ (resp. $\mathscr{L}(L, V)$) is here the grading defined by the intrinsic grading $\partial\lambda$ of the form λ (cf. (A.15)). As shown by (2.11), we have, for this *∂-grading* and $\xi \in L$; $a \in A^{\cdot}$ the operator grades $\partial\delta_0 = \partial(\varrho\wedge) = \partial\delta_\varrho = \partial(\Omega\wedge) = 0$ and $\partial\theta_0(\xi) = \partial\varrho(\xi) = \partial\theta_\varrho(\xi) = \partial i(\xi) = \partial\xi$; $\partial(a^{\cdot}) = \partial(\delta a\wedge) = \partial a^{\cdot}$ $\Lambda^*(L, V)$ possesses in addition the N-grading $\Lambda^* = \bigoplus_{n\in N}\Lambda^n(L, V)$ for which the grade of $\Omega\wedge$ is two and that of $\delta_0,\ \varrho\wedge,\ \delta_\varrho,\ \delta a\wedge$ one; that of $\theta_0(\xi),\ \varrho(\xi),\ \theta_\varrho(\xi),\ a^{\cdot}$ zero; that of $i(\xi)$, -1.

$$(\delta a \wedge \lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^{n+1} (-1)^{\beta_i + \partial a \partial \xi_i} (\xi_i\, a)\, \lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}), \qquad (2.15)[13]$$

$$(\Omega \wedge \lambda)(\xi_1, \ldots, \xi_{n+2}) = -\sum (-1)^{\alpha_{ij} + \partial\lambda(\partial\xi_i + \partial\xi_j)} \Omega(\xi_i, \xi_j)\, \lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{n+2}), \qquad (2.16)[14]$$

(iii) *$\Lambda_A^*(L, V)$ is stable under $i(\xi)$, $a^{\cdot}$, $\delta a \wedge$, $\Omega(\xi, \eta)$, $\theta_\varrho(\xi)$[15]; and for a local connection ϱ, under δ_ϱ and $\Omega \wedge$.*

Proof: (i) δ_0, $\varrho \wedge$ (hence δ_ϱ), $\theta_0(\xi)$, $\delta a \wedge$ and $\Omega \wedge$ have range in $\Lambda^*(E, V)$ since they are defined by means of the graded alternator. And $\varrho(\xi)\lambda$, $i(\xi)\lambda$, $a\lambda$ are graded alternate for λ graded alternate. The inclusions (2.11) are obvious, the intrinsic grade of an operator being the excess of the grade of its value over the sum of grades of its arguments.

(ii) Check of (2.12): we have $\Sigma_{n+1} = \bigcup_{1 \leqslant i < j \leqslant n+1} \Sigma_{ij} \circ \sigma_{ij}$ with σ_{ij} the permutation

$$\sigma_{ij} = \begin{pmatrix} 1 & 2 & 3 & \ldots & & & n+1 \\ i & j & \ldots & \hat{\imath} & \ldots \hat{\jmath} & \ldots & n+1 \end{pmatrix}, \quad 1 \leqslant i < j \leqslant n+1 \qquad (2.17)$$

and Σ_{ij} the subgroup of Σ_{n+1} leaving stable the subset $\{i, j\}$. Due to the graded antisymmetry of λ and the fact that $[\xi_j, \xi_i] = (-1)^{1 + \partial\xi_i \partial\xi_j} [\xi_i, \xi_j]$, we have $\sigma_{n+1} \circ \sigma_{ij} \lambda^\delta = \sigma_{ij} \lambda^\delta$ for $\sigma_{n+1} \in \Sigma_{ij}$.[16] Formula (2.10) then follows from the fact that $\operatorname{Card} \Sigma_{ij} = 2(n-1)!$ and $\chi(\xi, \sigma_{ij}) = -(-1)^{\alpha_{ij}}$.

Check (2.13): we have $\Sigma_{n+1} = \bigcup_{i \in I_{n+1}} \Sigma_i \circ \sigma_i$ with

$$\sigma_i = \begin{pmatrix} 1 & 2 & \ldots & & n+1 \\ i & 1 & \ldots & \hat{\imath} \ldots & n+1 \end{pmatrix}, \quad 1 \leqslant i \leqslant n+1 \qquad (2.18)$$

and Σ_i the subgroup of Σ_{n+1} leaving i invariant. We have $\sigma_{n+1} \circ \sigma_i \lambda^\varrho = \sigma_i \lambda^\varrho$ due to graded antisymmetry of λ. Formula (2.13) then follows from the facts that $\operatorname{Card} \Sigma_i = n!$ and

$$\chi(\xi, \sigma_i) = (-1)^{1 + i + \partial\xi_i(\partial\xi_1 + \ldots + \partial\xi_{i-1})}. \qquad (2.18a)$$

---

[13] With β_i as in (2.13).

[14] With α_{ij} as in (2.12).

[15] Observe that $\theta_\varrho(\xi)$ leaves $\Lambda_A^*(L, V)$ stable also for a non-local connection ϱ, this property is not explicitly stated in [1].

[16] See formula (B.7) of Appendix B.

Check of (2.14): we have now $\sigma_n \circ \sigma_i \lambda^\xi = \sigma_i \lambda^\xi$ for σ_n in the stabilizer Σ_i of i and

$$\sigma_i = \begin{pmatrix} 1 & 2 \ldots & & n \\ i & 2 \ldots & \hat{\imath} \ldots & n \end{pmatrix}, \qquad (2.19)$$

with $\chi(\xi, \sigma_i)$ as above and $\operatorname{Card} \Sigma_i = (n-1)!$. Thus

$$\begin{aligned} \{\theta_0(\xi)\lambda\}(\xi_1, \ldots, \xi_n) &= \sum_{i=1}^{n} (-1)^{1 + i + \partial\xi_i(\partial\xi_1 + \ldots + \partial\xi_{i-1}) + \partial\lambda\partial\xi} \times \\ &\quad \times \lambda([\xi, \xi_i], \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n) \\ &= \sum_{i=1}^{n} (-1)^{\partial\xi(\partial\xi_1 + \ldots + \partial\xi_{i-1} + \partial\lambda)} \times \\ &\quad \times \lambda([\xi_1, \ldots, \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, \ldots, \xi_n). \end{aligned} \qquad (2.20)$$

Check of (2.15): we have $\sigma_{n+1} \circ \sigma_i \lambda^a = \sigma_i \lambda^a$ for $\sigma_{n+1} \in \Sigma_i$ and σ_i as in (2.18). Formula (2.15) then follows from the already stated facts, $\operatorname{Card} \Sigma_i = n!$ and (2.18a).

Check of (2.16): we have $\sigma_{n+1} \circ \sigma_{ij} \lambda^\Omega = \sigma_{ij} \lambda^\Omega$ for σ_{ij} the permutation (2.17), where $n \to n+1$, and σ_{ij} in the subgroup Σ_{ij} of Σ_{n+2} leaving stable the subset $\{i, j\}$ (this because of the graded antisymmetry of λ and the fact that $\Omega(\eta, \xi) = -(-1)^{\partial\eta\partial\xi} \Omega(\xi, \eta)$. Formula (2.16) then follows from the facts that $\operatorname{Card} \Sigma_{ij} = 2(n!)$ and $\chi(\xi, \sigma_{ij}) = (-1)^{\alpha_{ij}+1}$.

(iii) Let $\lambda \in \Lambda_A^*(L, V)$; we have, from (2.7),

$$\begin{aligned} \{i(\xi)\lambda\}(\xi_1, \ldots, b\xi_k, \ldots, \xi_{n-1}) &= (-1)^{\partial\lambda\partial\xi} \lambda(\xi, \xi_1, \ldots, b\xi_k, \ldots, \xi_n) \\ &= (-1)^{\partial\lambda\partial\xi + \partial b\left(\partial\lambda + \sum_{i=1}^{k-1} \partial\xi_i\right)} b\lambda(\xi, \xi_1, \ldots, \xi_{n-1}) \\ &= (-1)^{\partial b\left[\partial(i(\xi)\lambda) + \sum_{i=1}^{k-1} \partial\xi_i\right]} b\,\{i(\xi)\lambda\}(\xi_1, \ldots, \xi_n), \end{aligned} \qquad (2.21)$$

further, from (2.8),

$$\begin{aligned} (a \cdot \lambda)(\xi_1, \ldots, b\xi_k, \ldots, \xi_n) &= (-1)^{\partial b\left(\partial\lambda + \sum_{i=1}^{k-1} \partial\xi_i\right)} ab\lambda(\xi_1, \ldots, \xi_n) \\ &= (-1)^{\partial b\left(\partial(a\cdot\lambda) + \sum_{i=1}^{k-1} \partial\xi_i\right)} b(a \cdot \lambda)(\xi_1, \ldots, \xi_n), \end{aligned} \qquad (2.22)$$

further, from (2.9) using (1.5), since V is a left A-module

$$\lambda^a(b\xi_1, \ldots, \xi_{n+1}) = (-1)^{(\partial b+\partial\xi_1)(\partial a+\partial\lambda)} b(\xi_1 a)\lambda(\xi_2, \ldots, \xi_{n+1})$$

$$= (-1)^{\partial b\,\partial\lambda^a} b\lambda^a(\xi_1, \ldots, \xi_n) \tag{2.23}$$

and for $k \geqslant 1$

$$\lambda^a(\xi_1, \ldots, b\xi_k, \ldots, \xi_n) = (-1)^{\partial\xi_1(\partial a+\partial\lambda)+\partial b\left(\partial\lambda+\sum_{i=2}^{k-1}\partial\xi_i\right)}(\xi_1 a)b\lambda(\xi_2, \ldots, \xi_n)$$

$$= (-1)^{\partial b\left(\partial\lambda^a+\sum_{i=1}^{n-1}\partial\xi_i\right)} b\lambda^a(\xi_1, \ldots, \xi_n), \tag{2.24}$$

hence λ^a is A-linear, and so is $\delta a\wedge$, since A_{n+1} obviously commutes with multiplication from the left by b.

To prove that δ_ϱ and $\theta_\varrho(\xi)$ leave $\Lambda_A^*(L, V)$ stable, we shall use the intertwining Theorem 3.6 of the next section.[17] We denote by U an arbitrary graded commutative algebra, and observe that, by using Proposition 3.3, $\delta_\varrho\lambda$ is A-linear if the restriction $(\delta_\varrho\lambda)_U^0$ of $(\delta_\varrho\lambda)_U$ to $(L_U)^0$ is $(A_U)^0$-linear. Now, using (3.27) (cf. Proposition 3.6) we may write

$$\delta_\varrho(\lambda_U^0) = (\delta_\varrho\lambda)_U^0 = \delta_\varrho\lambda_U^0, \tag{2.25}$$

where λ_U^0 is the restriction of λ_U to L_U^0.

The A_U^0-linearity of the l.h.s of (2.25) follows now by the Proposition and Remark 3.5 and Theorem 1.8(iv) of [1]. The same argument proves A-linearity of $\delta_0\lambda$, and A-linearity of $\varrho\wedge\lambda$ follows by difference. We use the same trick to prove A-linearity of $\theta_\varrho(\xi)$.

By (3.30) we have

$$\theta_\varrho(u\otimes\xi)\lambda_U = (u\otimes id)\big(\theta_\varrho(\xi)\lambda\big)_U \tag{2.26}$$

for all $\xi\in L$ and $u\in U$. Now, by Theorem 1.8 (iv) of [1] the l.h.s, when $u\otimes\xi$ as well as the arguments of λ_U are restricted to L_U^0, is A_U^0-linear. This implies A_U^0-linearity of the r.h.s., and this in turn, owing to the arbitrariness of U and $u\in U$, implies A-linearity of $\theta_\varrho(\xi)\lambda$ by the same arguments as used in Proposition 3.3. The A-linearity of $\theta_0(\xi)\lambda$ and $\varrho(\xi)\lambda$ follows.

---

[17] Our reader may skip the rest of this proof and wait until Remark 2.5 (ii) for an alternative proof.

2.3. *Remark*. As a particular case of an A-module V we can take $V = A$. A local A-connection is then given by

$$d(\xi) = \xi \tag{2.27}$$

assigning to each $\xi\in L$ its action on A as in the definition of the Lie–Cartan pair. The covariant derivative acting on $\Lambda^*(L, A)$ corresponding to this connection is denoted simply δ. Thus (2.1)–(2.3) and (2.27) imply

$$(\delta\lambda)(\xi_1, \ldots, \xi_{n+1}) = \sum_{1\leqslant i<j\leqslant n+1}(-1)^{\alpha_{ij}}\lambda([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{n+1}) +$$
$$+(-1)^{\beta_i}\xi_i\{\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\} \tag{2.28}$$

for $\lambda\in\Lambda^n(L, A)$. The notation $\delta a\wedge$ for the operator (2.9) agrees then with (2.28) and with the exterior product notation of Section 4.

2.4. THEOREM. *Let (L, A) be a unital graded Lie–Cartan pair, let V be a unital graded left A-module, let ϱ be a V-connection for (L, A), let $a, b\in A$ and $\xi, \eta\in L$. The classical operators defined in Definition 2.1 fulfil the following relations, where the brackets $[\ ,\ ]$ denote graded commutators:*[18]

$$[\delta_0, a\cdot] = [i(\xi), a\cdot] = [\theta_0(\xi), a\cdot] = [b\cdot, a\cdot] = [\delta b\wedge, a\cdot] = 0, \tag{2.31}$$

$$[\varrho(\xi), a\cdot] = (\xi a), \tag{2.32}$$

$$[\varrho\wedge, a\cdot] = \delta a\wedge, \tag{2.33}$$

$$[\theta_\varrho(\xi), a\cdot] = (\xi a), \tag{2.34}$$

$$[\delta_\varrho, a\cdot] = \delta a\wedge, \tag{2.35}$$

$$i(\xi)i(\eta)+(-1)^{\partial\xi\partial\eta}i(\eta)i(\xi) = 0, \tag{2.36}$$

$$[\theta_\varrho(\xi), \theta_\varrho(\eta)] = \theta_\varrho([\xi, \eta])+\Omega(\xi, \eta), \tag{2.37}$$

$$\delta_\varrho^2 = \Omega\wedge, \tag{2.38}$$

$$\delta_\varrho i(\xi)+i(\xi)\delta_\varrho = \theta_\varrho(\xi), \tag{2.39}$$

$$[i(\xi), \theta_\varrho(\eta)] = i([\xi, \eta]), \tag{2.40}$$

$$[\theta_\varrho(\xi), \delta_\varrho] = [i(\xi), \Omega\wedge], \tag{2.41}$$

$$\delta_0^2 = 0, \tag{2.38a}$$

$$\delta_0 i(\xi)+i(\xi)\delta_0 = \theta_0(\xi), \tag{2.39a}$$

---

[18] Graded commutators of operators on $\Lambda^*(L, V)$ w.r.t. the grading of the latter defined by the intrinsic grading $\partial\lambda$ of the form λ.

$$[i(\xi),\, \theta_0(\eta)] = i([\xi,\, \eta]), \tag{2.40a}$$

$$[\theta_0(\xi),\, \delta_0] = 0, \tag{2.41a}$$

$$[i(\xi),\, \varrho(\eta)] = 0. \tag{2.42}$$

Proof: Check of (2.31): δ_0 evidently commutes with a, as acting "internally" on the arguments of λ without altering $\partial\lambda$. Further, we have

$$\begin{aligned}\{[i(\xi),\, a\cdot]\,\lambda\}(\xi_1, \ldots, \xi_{n-1}) &= \{(-1)^{(\partial\lambda+\partial a)\partial\xi} - (-1)^{\partial i(\xi)\partial(a\cdot)+\partial\lambda\partial\xi}\, a\lambda(\xi, \xi_1, \ldots, \xi_{n-1})\\ &= \{(-1)^{(\partial\lambda+\partial a)\partial\xi} - (-1)^{\partial\xi\partial a+\partial\lambda\partial\xi}\}\, a\lambda(\xi, \xi_1, \ldots, \xi_{n-1}).\end{aligned} \tag{2.43}$$

Analogously

$$\begin{aligned}&\{(a\cdot\lambda)^{\xi} - (-1)^{\partial\theta_0(\xi)\partial(a\cdot)}\, a\cdot\lambda^{\xi}\}(\xi_1, \ldots, \xi_n)\\ &\quad= \{(-1)^{(\partial\lambda+\partial a)\partial\xi} - (-1)^{\partial\xi\partial a+\partial\lambda\partial\xi}\}\, a\lambda([\xi, \xi_1], \xi_2, \ldots, \xi_n),\end{aligned} \tag{2.44}$$

whence one obtains the vanishing of the ∂-graded commutator $[\theta_0(\xi),\, a\cdot]$, since A_n obviously commutes with $a\cdot$.

Further, we have obviously $[a\cdot, b\cdot] = [a, b] = 0$; on the other hand, from (2.15)

$$\begin{aligned}&\{\delta b \wedge (a\cdot\lambda)\}(\xi_1, \ldots, \xi_{n+1})\\ &\quad= \sum_{i=1}^{n+1} (-1)^{1+i+\partial\xi_i(\partial\lambda+\partial a+\sum_{k=1}^{i-1}\partial\xi_k)+\partial b\partial\xi_i}\, (\xi_i b)\, a\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\\ &\quad= (-1)^{\partial a\partial b} \sum_{i=1}^{n+1} (-1)^{1+i+\partial\xi_i(\partial\lambda+\sum_{k=1}^{i-1}\partial\xi_k)+\partial b\partial\xi_i}\, a(\xi_i b)\, \lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\\ &\quad= (-1)^{\partial a\partial b}\{a\cdot(\delta b \wedge \lambda)\}(\xi_1, \ldots, \xi_n).\end{aligned} \tag{2.45}$$

Check of (2.32): we have from (2.5), (2.8), using (1.9)

$$\begin{aligned}\varrho(\xi)(a\cdot\lambda)(\xi_1, \ldots, \xi_n) &= \varrho(\xi)\{a\lambda(\xi_1, \ldots, \xi_n)\}\\ &= (-1)^{\partial a\partial\xi}\, a\varrho(\xi)\,\lambda(\xi_1, \ldots, \xi_n) + (\xi a)\,\lambda(\xi_1, \ldots, \xi_n).\end{aligned} \tag{2.46}$$

From (2.32) and $[\theta_0(\xi),\, a\cdot] = 0$ follows (2.34) (cf. (2.6)).

Check of (2.33): we have from (2.13), (2.15), (2.8), again using (1.9)

$$\begin{aligned}\{\varrho \wedge (a\cdot\lambda)\}(\xi_1, \ldots, \xi_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{\beta_i+\partial a\partial\xi_i}\, \varrho(\xi_i)\{a\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\}\\ = \sum_{i=1}^{n+1} (-1)^{\beta_i}\{a\varrho(\xi_i)\,\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}) &+ (-1)^{\partial a\partial\xi_i}(\xi_i a)\,\lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\\ = \{a\varrho + \delta a \wedge \lambda\}(\xi_1, \ldots, \xi_{n+1}).\end{aligned} \tag{2.47}$$

From (2.33) and $[\delta_0,\, a\cdot] = 0$ follows (2.35) (cf. (2.3)).

Check of (2.36): we have from (2.27), for $n \geqslant 2$

$$\begin{aligned}\{i(\xi)\,i(\eta)\,\lambda\}(\xi_1, \ldots, \xi_{n-2}) &= (-1)^{\partial\xi(\partial\eta+\partial\lambda)}\{i(\eta)\,\lambda\}(\xi, \xi_1, \ldots, \xi_{n-2})\\ &= (-1)^{\partial\xi\partial\eta+\partial\lambda(\partial\xi+\partial\eta)}\,\lambda(\eta, \xi, \xi_1, \ldots, \xi_{n-2}),\end{aligned} \tag{2.48}$$

which, by the graded symmetry of λ, changes by a factor $(-1)^{\partial\xi\partial\eta+1}$ upon exchange of ξ and η.

The rest of our proof is based on the intertwining Theorem 3.6 of the next section. The proofs of the latter are independent of the present section. We begin with the proof of (3.28) which most simply shows how intertwining yields a proof: with U an arbitrary graded commutative algebra, we have for $\lambda \in \Lambda^*(L, V)$ from (3.30), (3.34)

$$(\delta_\varrho^2 - \Omega\wedge)(\lambda_U) = \{(\lambda_\varrho - \Omega\wedge)\,\lambda\}_U. \tag{2.49}$$

Now, since the pair (L_U^0, A_U^0) obtained from the zero grade parts of L_U and A_U is a Lie–Cartan pair in the sense of [1], we know from Theorem 1.8 formula (52) there, that the l.h.s. of (2.49) vanishes in the restriction to $V_U^0 \times V_U^0 \times \ldots \times V_U^0$. It then follows by the last assertion in Proposition 3.3 that $(\delta_\varrho - \Omega\wedge)\,\lambda$ vanishes.

Proof of (2.37): We have, for $u, v \in U$ fulfilling $\partial u = \partial\xi$, $\partial v = \partial\eta$, from (3.30)

$$\begin{aligned}\theta_\varrho(u\otimes\xi)\,\theta_\varrho(v\otimes\eta)(\lambda_U) &= \theta_\varrho(u\otimes\xi)(v\otimes id)\{\theta_\varrho(\eta)\,\lambda\}_U\\ &= (v\otimes id)(u\otimes id)\{\theta_\varrho(\xi)\,\theta_\varrho(\eta)\,\lambda\}_U\\ &= (vu\otimes id)\{\theta_\varrho(\xi)\,\theta_\varrho(\eta)\,\lambda\}_U,\end{aligned} \tag{2.50}$$

where we used the fact that $\theta_\varrho(u\otimes\xi)$ commutes with $v\otimes id$; this is obviously the case for $\theta_0(u\otimes\xi)$ which acts "internally", with γ_i independent of $\partial\lambda$, cf. (2.14); but this holds also for $\varrho(u\otimes\xi)$. We have indeed, from (3.14) and (A.7)

$$\begin{aligned}\varrho(u\otimes\xi)(v\otimes id) &= \{u\otimes\varrho(\xi)\}(v\otimes id)\\ &= (-1)^{\partial v\partial\xi}\, uv\otimes\varrho(\xi)\\ &= (-1)^{\partial v(\partial u+\partial\xi)}\, vu\otimes\varrho(\xi),\end{aligned} \tag{2.51}$$

thus

$$\begin{aligned}\varrho(u\otimes\xi)(v\otimes id) &= (-1)^{\partial v(\partial u+\partial\xi)}(v\otimes id)\,\varrho(u\otimes\xi),\\ \theta_\varrho(u\otimes\xi)(v\otimes id) &= (-1)^{\partial v(\partial u+\partial\xi)}(v\otimes id)\,\theta_\varrho(u\otimes\xi).\end{aligned} \tag{2.52}$$

We have on the other hand, from (2.37), (3.4) and (A.8)

$$\begin{aligned}\theta_\varrho([u\otimes\xi,\, v\otimes\eta])(\lambda_U) &= (-1)^{\partial\eta\partial\xi}\,\theta_\varrho(uv\otimes[\xi, \eta])(\lambda_U)\\ &= (vu\otimes id)\{\theta_\varrho([\xi, \eta])\,\lambda\}_U,\end{aligned} \tag{2.53}$$

hence, from this, (2.50) and (3.35)

$$\{\theta_\varrho(u\otimes\xi)\,\theta_\varrho(v\otimes\eta)-\theta_\varrho(v\otimes\eta)\,\theta_\varrho(u\otimes\xi)-$$
$$=\theta_\varrho([u\otimes\xi,\ v\otimes\eta])-\Omega(u\otimes\xi,\ v\otimes\eta)\}(\lambda_U)$$
$$=(vu\otimes id)\,\{([\theta_\varrho(\xi),\ \theta_\varrho(\eta)]-\theta_\varrho([\xi,\ \eta])-\Omega(\xi,\ \eta))\,\lambda\}_U. \quad (2.54)$$

The l.h.s. now vanishes in the restriction to $V_U^0\times\ldots\times V_U^0$ by formula (51) in [1], hence the last $\{\ \}$ vanishes by the above procedure.

Proof of (2.39): We have, from (3.27), (3.31) and (3.30)

$$\{\delta_\varrho\, i(u\otimes\xi)+i(u\otimes\xi)\,\delta_\varrho-\theta_\varrho(u\otimes\xi)\}(\lambda_U)$$
$$=\delta_\varrho(u\otimes id)\,\{i(\xi)\,\lambda\}_U+(u\otimes id)\,\{(i(\xi)\,\delta_\varrho+\theta_\varrho(\xi))\,\lambda\}_U$$
$$=(u\otimes id)\,\{(\delta_\varrho\, i(\xi)+i(\xi)\,\delta_\varrho-\theta_\varrho(\xi))\,\lambda\}_U. \quad (2.55)$$

Now formula (53) of [1] entails the vanishing of the l.h.s. whence the vanishing of the last $\{\ \}$.

Proof of (2.40): We have, from (3.30), (3.31), for $u,\ v\in U$ with $\partial u=\partial\xi,\ \partial v=\partial\eta$, since then, by (A.8): $[u\otimes\xi,\ v\otimes\eta]=(-1)^{\partial\xi\partial\eta}\,uv\otimes[\xi,\ \eta]$

$$\{i(u\otimes\xi)\,\theta_\varrho(v\otimes\eta)-\theta_\varrho(v\otimes\eta)\,i(u\otimes\xi)-i([u\otimes\xi,\ v\otimes\eta])\}(\lambda_U)$$
$$=i(u\otimes\xi)(v\otimes id)\,\{\theta_\varrho(\eta)\,\lambda\}_U-\theta_\varrho(v\otimes\eta)(u\otimes id)\,\{i(\xi)\,\lambda\}_U-$$
$$-(-1)^{\partial\xi\partial\eta}\,i(uv\otimes[\xi,\ \eta])(\lambda_U)$$
$$=(v\otimes id)(u\otimes id)\,\{i(\xi)\,\theta_\varrho(\eta)\,\lambda\}_U-(u\otimes id)(v\otimes id)\,\{\theta_\varrho(\eta)\,i(\xi)\,\lambda\}_U-$$
$$-(-1)^{\partial\xi\partial\eta}(uv\otimes id)\,\{i([\xi,\ \eta])\,\lambda\}_U$$
$$=(vu\otimes id)\,\{(i(\xi)\,\theta_\varrho(\eta)-(-1)^{\partial\xi\partial\eta}\,\theta_\varrho(\eta)\,i(\xi)-i([\xi,\ \eta]))\,\lambda\}_U. \quad (2.56)$$

Observe that we may commute $\theta_\varrho(v\otimes\eta)$ and $u\otimes id$ by (2.52); and $i(u\otimes\xi)$ and $v\otimes id$ because, due to (2.7):

$$i(u\otimes\xi)(v\otimes id)=(-1)^{\partial v(\partial u+\partial\xi)}\,(v\otimes id)\,i(u\otimes\xi).$$

We then conclude the vanishing of the last $\{\ \}$ in (3.56) from the vanishing of the l.h.s. in restriction to $V_U^0\times\ldots\times V_U^0$, as implied by formula (54) in [1].

We have from (2.38) and (2.39)

$$[\theta_\varrho(\xi),\ \delta_\varrho]=\theta_\varrho(\xi)\,\delta_\varrho-\delta_\varrho\,\theta_\varrho(\xi)=i(\xi)\,\delta_\varrho^2+\delta_\varrho\, i(\xi)\,\delta_\varrho-$$
$$-\delta_\varrho\, i(\xi)\,\delta_\varrho-\delta_\varrho^2\, i(\xi)=[i(\xi),\ \Omega\wedge]. \quad (2.57)$$

Formulae (2.38a) through (2.41a) follow from the corresponding formulae (2.38) through (2.41) by replacing the graded Lie–Cartan pair $(L,\ A)$ by its depletion

$(L,\ \mathbf{R})$ (or $(L,\ \mathbf{C})$), and then making the choice $\varrho=0$. As for (2.42) it follows from (2.40), (2.40a) due to (2.6).

2.5. *Remark*. The proof of the statements in Theorem 2.4 and in Proposition 2.2 (iii) can either be inferred from the corresponding results of the commuting case (Theorem 1.8 of [1]) by tensorizing by U as in Proposition 3.5; or else verified directly, which we did here when verification is not complicated. Of course, the first method could be used all through, e.g. (2.32), resp. (2.36) could be inferred from the equalities

$$\{[\varrho(u\otimes\xi),\ (v\otimes a)\cdot]-(u\otimes\xi)(v\otimes a)\cdot\}(\lambda_U)=(vu\otimes id)\,\{[\varrho(\xi),\ a\cdot]-(\xi a)]\,\lambda\}_U, \quad (2.58)$$

resp.

$$\{i(u\otimes\xi)\,i(v\otimes\eta)+i(v\otimes\eta)\,i(u\otimes\xi)\}(\lambda_U)$$
$$=(vu\otimes id)\,\{(i(\xi)\,i(\eta)+(-1)^{\partial\eta\partial\xi}\,i(\eta)\,i(\xi))\,\lambda\}_U \quad (2.59)$$

valid when $\partial u=\partial\xi,\ \partial v=\partial\eta$.

As for the fact that δ_ϱ and $\theta_\varrho(\xi)$ leave $\Lambda_A^*(L,\ V)$ stable, we gave a proof based on reduction to the commutative case, since verification is cumbersome for δ_ϱ. However, one could more simply proceed as follows: Let $\lambda\in\Lambda_A^n(L,\ V)$:

(i) Let $b\in A$ and $1\leqslant k\leqslant n$; we have, using (1.9)

$$\{\varrho(\xi)\,\lambda\}(\xi_1,\ \ldots,\ b\xi_k,\ \ldots,\ \xi_n)$$
$$=(-1)^{\partial b(\partial\lambda+\sum_{i=1}^{k-1}\partial\xi_i)}\,\{((-1)^{\partial b\partial\xi}\,b\varrho(\xi)+(\xi b))\,\lambda\}(\xi_1,\ \ldots,\ \xi_n). \quad (2.60)$$

On the other hand, with $\theta_0^j(\xi)\,\lambda$ the jth term on the r.h.s. of (2.14) we have, by virtue of $[\xi,\ b\xi_j]=(-1)^{\partial b\partial\xi}\,b\,[\xi,\ \xi_i]+(\xi b)\,\xi_j$ the equality

$$\{\theta_0(\xi)\,\lambda\}(\xi_1,\ \ldots,\ b\xi_k,\ \ldots,\ \xi_n)$$
$$=\sum_{j=1}^{n}(-1)^{\partial b(\partial\xi+\partial\lambda+\sum_{i=1}^{k-1}\partial\xi_k)}\,\{(b\theta_0^j(\xi)-\delta_{jk}(-1)^{\partial b\partial\xi}(\xi b))\,\lambda\}(\xi_1,\ \ldots,\ \xi_n) \quad (2.61)$$

yielding the sum

$$\{\theta_0(\xi)\,\lambda\}(\xi_1,\ \ldots,\ b\xi_k,\ \ldots,\ \xi_n)$$
$$=(-1)^{\partial b(\partial\xi+\partial\lambda+\sum_{i=1}^{k-1}\partial\xi_i)}\,\{(b\theta_0(\xi)-(-1)^{\partial b\partial\xi}(\xi b))\,\lambda\}(\xi_1,\ \ldots,\ \xi_n). \quad (2.62)$$

A-linearity of $\theta_\varrho(\xi)\,\lambda=\theta_0(\xi)\,\lambda+\varrho(\xi)\,\lambda$ follows by performing summation.

(ii) We proceed by induction w.r.t. n: assume that we proved that $\delta_\varrho \lambda$ belongs to $\Lambda^n_A(L, V)$ if $\lambda \in \Lambda^{n-1}_A(L, V)$: with $\lambda \in \Lambda^n_A(L, V)$, we have, by what precedes, that $i(\xi)\delta_o \lambda = \theta_\varrho(\xi)\lambda - \delta_\varrho i(\xi)\lambda$ belongs to $\Lambda^n_A(L, V)$ for all $\xi \in L$, i.e. we have

$$\begin{aligned}\{i(\xi)\delta_\varrho \lambda\}(b\xi_1, \xi_2, \ldots, \xi_{n-1}) &= (-1)^{\partial\xi\partial\lambda}(\delta_\varrho \lambda)(\xi, b\xi_1, \xi_2, \ldots, \xi_{n-1}) \\ &= (-1)^{\partial b(\partial\lambda + \partial\xi)} b\delta_\varrho \lambda(\xi, \xi_1, \ldots, \xi_{n-1}).\end{aligned} \tag{2.63}$$

We proved A-linearity of $\delta_\varrho \lambda$ w.r.t. its second argument, implying multi A-linearity, since λ is graded symmetric. Now for $\lambda \in \Lambda^0(L, V) = V$, $(\delta_\varrho \lambda)(b\xi_1) = \varrho(b\xi_1)\lambda = b\delta_\varrho(\xi_1)\lambda = b\delta_\varrho \lambda(\xi_1)$ if ϱ is local.

The next proposition shows how the classical operators acting on $\Lambda^*(L, V)$ are obtained from the classical operators acting on $\Lambda^*(L, A)$.

2.6. Proposition. *Let (L, A), V, and ϱ be as in Definition* 2.1, *V finitely generated and projective, with δ_0, $\varrho\wedge$, $\theta_0(\xi)$, $\varrho(\xi)$, $\theta_\varrho(\xi)$, $\Omega(\xi, \eta)$, $i(\xi)$ acting on $\Lambda^n(L, V)$ as defined there. And consider on the other hand A itself as a left A-module with the A-connection $d = id$: $\xi \in L \mapsto \xi$ (see* (2.27)) *and the corresponding classical operators δ_0, $\wedge$, $\delta = \delta_d$, $\theta_0(\xi)$, $d(\xi) = \xi$, $\theta_d(\xi) = \theta(\xi)$, $\Omega(\xi, \eta)$, $i(\xi)$ acting on $\Lambda^n(L, A)$. Furthermore, we consider the left A-module V as a right A-module (cf.* (A.42) *in Appendix* A), *this yielding the isomorphism (see* A)

$$\Lambda^n(L, V) = V \otimes_A \Lambda^n(L, A), \tag{2.64}$$

with the identification

$$\begin{aligned}&(X \otimes \alpha)(\xi_1, \ldots, \xi_n) = X\alpha(\xi_1, \ldots, \xi_n), \\ &X \in V, \quad \alpha \in \Lambda^n(L, A), \quad \xi_1, \ldots, \xi_n \in L\end{aligned} \tag{2.64a}$$

and the ensuing fact that $\Lambda^*(L, V)$ becomes a right $\Lambda^*(L, A)$-module, by setting

$$(X \otimes \alpha)\beta = X \otimes (\alpha \wedge \beta), \qquad X \in V,\ \alpha, \beta \in \Lambda^*(L, A). \tag{2.65}$$

We then have the following expressions for the classical operators on $\Lambda^*(L, V)$ in terms of those on $\Lambda^*(L, A)$: for $\xi, \eta \in L^\cdot$, $X \in V^\cdot$, $\alpha \in \Lambda^*(L, A)$ we have

$$\varrho \wedge (X \otimes \alpha) = (\varrho \wedge X)\alpha + X \otimes (d \wedge \alpha), \tag{2.66}$$

$$\delta_0(X \otimes \alpha) = X \otimes \delta_0 \alpha, \tag{2.67}$$

$$\delta_\varrho(X \otimes \alpha) = (\delta_\varrho X)\alpha + X \otimes \delta\alpha, \tag{2.68}$$

$$\theta_0(\xi)(X \otimes \alpha) = (-1)^{\partial\xi\partial X} X \otimes \{\theta_0(\xi)\alpha\}, \tag{2.69}$$

$$\varrho(\xi)(X \otimes \alpha) = \{\varrho(\xi)X\} \otimes \alpha + (-1)^{\partial\xi\partial X} X \otimes (\xi\alpha), \tag{2.70}$$

$$\theta_\varrho(\xi)(X \otimes \alpha) = \{\theta_\varrho(\xi)X\} \otimes \alpha + (-1)^{\partial\xi\partial X} X \otimes \{\theta(\xi)\alpha\}, \tag{2.71}$$

$$i(\xi)(X \otimes \alpha) = (-1)^{\partial\xi\partial X} X \otimes \{i(\xi)\alpha\}, \tag{2.72}$$

$$\Omega(\xi, \eta)(X \otimes \alpha) = \{\Omega(\xi, \eta)X\} \otimes \alpha. \tag{2.73}$$

Proof: Check of (2.66): we have, using (B.43)

$$\begin{aligned}\{(\varrho \wedge X)\alpha\}(\xi_1, \ldots, \xi_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1+\partial\xi_i\left(\partial\alpha + \sum_{k=1}^{i-1}\partial\xi_k\right)} \times \\ \times (-1)^{\partial\xi_i\partial X}\{\varrho(\xi_i)X\}\alpha(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\end{aligned} \tag{2.74}$$

and furthermore

$$\begin{aligned}&\{X(id \otimes \alpha)\}(\xi_1, \ldots, \xi_{n+1}) \\ &\quad = \sum_{i=1}^{n+1} (-1)^{i+1+\partial\xi_i\left(\partial\alpha + \sum_{k=1}^{i-1}\partial\xi_k\right)} X\xi_i\{\alpha(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1})\},\end{aligned} \tag{2.75}$$

whence (2.66), using the fact that

$$\varrho(\xi)(Xa) = \{\varrho(\xi)X\}a + (-1)^{\partial\xi\partial X}X(\xi a), \qquad X = V^\cdot,\ \xi \in L^\cdot,\ a \in A. \tag{2.76}$$

Definition (2.12) immediately implies (2.67).

Adding (2.66) and (2.67), one gets (2.68). On the other hand, (2.69) immediately follows from definition (2.14).

Check of (2.70): we have, for $\xi_1, \ldots, \xi_n \in L$, using (2.64a) and (2.76)

$$\begin{aligned}\{\varrho(\xi)(X \otimes \alpha)\}(\xi_1, \ldots, \xi_n) &= \varrho(\xi)\{X\alpha(\xi_1, \ldots, \xi_n)\} \\ &= \{\varrho(\xi)X\}\alpha(\xi_1, \ldots, \xi_n) + (-1)^{\partial\xi\partial X}X\xi\{\alpha(\xi_1, \ldots, \xi_n)\}.\end{aligned} \tag{2.77}$$

Adding (2.69) and (2.70), one gets (2.71).

The definitions (2.7) and (2.5) immediately imply (2.72) and (2.73).

3. Tensorization by graded commutative algebras

Given a unital graded Lie–Cartan pair (L, A) together with a graded commutative algebra U, we now show that the skew tensor products[19]

$$\begin{aligned}L_U &= U \otimes L, \\ A_U &= U \otimes A\end{aligned} \tag{3.1}$$

naturally yield a graded Lie–Cartan pair (L_U, A_U) in such a way that

(i) each V-connection ϱ of (L, A) naturally yields a V_U-connection of (L_U, A_U)

[19] Of graded algebras, cf. Appendix, (A.2). L_U is thus a Lie super algebra and A_U a graded commutative algebra.

for the unital graded left A_U-module

$$V_U = U \otimes V, \tag{3.2}$$

(ii) each $\lambda \in \Lambda^n_A(L, V)$ extends to a $\lambda_U \in \Lambda^n_{A_U}(L_U, V_U)$, the map $\lambda \mapsto \lambda_U$ intertwining the classical operators.

We first describe the module E_U, E a unital graded left A-module.

3.1. Lemma. *Let A and U be real (complex) unital graded-commutative algebras, let E be a unital graded left A-module. By letting the skew tensor product $A_U = U \otimes A$ act on*[20]

$$E_U = U \otimes E \tag{3.3}$$

as

$$(u \otimes a)(v \otimes X) = (-1)^{\partial a \partial v}\, uv \otimes aX \qquad u \in U,\ v \in U^{\cdot},\ a \in A^{\cdot},\ X \in V \tag{3.4}$$

we make E_U into a unital graded left A_U-module.

Proof: The grade of the r.h.s. of (3.4) is $\partial u + \partial v + \partial a + \partial X = \partial(u \otimes a) + \partial(v \otimes X)$. On the other hand, (3.4) obviously specifies an E_U-valued bilinear product such that

$$\alpha(\beta X) = (\alpha\beta) X, \qquad \alpha, \beta \in A_U,\ X \in E_U \tag{3.5}$$

by (A.7) (cf. footnote 2 in Appendix A).

3.2. *Remark*. The left A_U-module E_U can be obtained as the tensor product

$$E_U = A_U \otimes_A E \tag{3.6}$$

of the left A_U-, right A-, bimodule A_U by the left A-module E.

Proof: Indeed, one has $U \otimes A \otimes_A E \sim U \otimes E$ with $u \otimes a \otimes X = u \otimes aX$, $u \in U$, $a \in A$, $X \in E$. And, with $v \in U$, $u \in V^{\cdot}$, $b \in A$

$$\begin{aligned}(v \otimes b)(u \otimes aX) &= (v \otimes b)\{u \otimes a \otimes X\} = \{(v \otimes b)(u \otimes a)\} \otimes X \\ &= (-1)^{\partial b \partial u}(vu \otimes ba) \otimes X = (-1)^{\partial b \partial u}\, vu \otimes baX.\end{aligned} \tag{3.7}$$

We now describe the map $\lambda \mapsto \lambda_U$ acting on $\Lambda^n_A(E, F)$, E, F unital graded left A-modules, A, U unital graded commutative algebras. In order to motivate definition (3.9) to follow, let us note that, with μ a F_U-valued A_U-n-linear form on E_U, and for $u_i \in U'$, $\xi_i \in E'$ writing $u_i \otimes \xi_i = (u_i \otimes 1)(1 \otimes \xi_i)$, A_U-n-linearity implies

$$\mu(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n) = (-)^{\sum_{1 \le i < j \le n} \partial\xi_i \partial u_j}(u_1 \ldots u_n \otimes 1)\,\mu(1 \otimes \xi_1, \ldots, 1 \otimes \xi_n). \tag{3.8}$$

[20] Unless specified, all tensor products are over $\boldsymbol{R}$ (over $\boldsymbol{C}$). We recall that we denote by E' the set $E^0 \cup E^1$ of homogeneous elements of the graded vector space E; and that we defined the tensor product of graded vector spaces as a graded vector space.

3.3. Proposition. *Let A and U be unital graded commutative algebras and let E and F be unital graded left A-modules.*[21] *Then defining λ_U as follows for $\lambda \in \mathscr{L}^n(E, F)$*

$$\lambda_U(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n) = (-1)^{\partial\lambda \sum_{i=1}^n \partial u_i + \sum_{1 \le i < j \le n} \partial\xi_i \partial u_j}\, u_1 \ldots u_n \otimes \lambda(\xi_1, \ldots, \xi_n) \tag{3.9}$$

we define a map $\lambda \mapsto \lambda_U$ from $\mathscr{L}^n(E, F)$ to $\mathscr{L}^n(E_U, F_U)$ preserving grade and commuting with the σ_n, $\sigma \in \Sigma_n$ and yielding a homomorphism:[22] *$\Lambda^n_A(E, F) \to \Lambda^n_{A_U}(E_U, F_U)$.*

Furthermore, for any choice of $\alpha_i \in \boldsymbol{Z}/2$, $i = 1, \ldots, n$ the specification of the restrictions of λ_U to $E_U^{(\alpha_1)} \times \ldots \times E_U^{(\alpha_n)}$ for all choices of U determines the form λ. Moreover, for λ to be A-linear it is enough that the above restrictions are $(A_U)^0$-linear.

Proof: The grade of the r.h.s. of (3.9) is $\partial\lambda + \sum_{i=1}^n (\partial u_i + \partial\xi_i)$: thus $\partial\lambda_U = \partial\lambda$.

We check that $\lambda_U \in \mathscr{L}_{A_U}(E_U, F_U)$ if $\lambda \in \mathscr{L}_A(E, F)$: we have by (3.9), for $u \in U^s$, $a_n \in A^p$, using the short hands $\partial u_i = s_i$, $\partial\xi_i = \alpha_i$, $\varepsilon = (-1)^{\partial\lambda \sum_{i=1}^n \partial u_i + \sum_{1 \le i < j \le n} \partial\xi_i \partial u_j}$ and taking account of the fact that $(u \otimes a)(u_k \otimes \xi_k) = (-1)^{ps_k}\, uu_k \otimes a\xi_k$

$$\begin{aligned}
&\lambda_U(u_1 \otimes \xi_1, \ldots, (u \otimes a)(u_k \otimes \xi_k), \ldots, u_n \otimes \xi_n) \\
&= \varepsilon(-1)^{ps_k + s(\alpha_1 + \ldots + \alpha_{k-1}) + p(s_{k+1} + \ldots + s_n) + s\partial\lambda}\, u_1 \ldots (uu_k) \ldots u_n \otimes \lambda(\xi_1, \ldots, a\xi_k, \ldots, \xi_n) \\
&= \varepsilon(-1)^{p(s_k + \ldots + s_n) + s(\alpha_1 + \ldots + \alpha_{k-1}) + s\partial\lambda + s(s_1 + \ldots + s_{k-1}) + p(\alpha_1 + \ldots + \alpha_k)} \times \\
&\qquad \times uu_1 \ldots u_n \otimes a\lambda(\xi_1, \ldots, \xi_n) \\
&= \varepsilon(-1)^{(s+p)\partial\lambda + (s+p)(s_1 + \ldots + s_{k-1} + \alpha_1 + \ldots + \alpha_k)}(u \otimes a)\{u_1 \ldots u_n \otimes \lambda(\xi_1, \ldots, \xi_n)\} \\
&= (-1)^{(s+p)\left(\partial\lambda + \sum_{i=1}^{k-1} s_i + \alpha_i\right)}(u \otimes a)\,\lambda_U(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n).
\end{aligned} \tag{3.10}$$

[21] Both real or complex, as well as E, F.

[22] I.e. a grade-zero homomorphism of the graded vector space (in fact for the right A-module) structure of $\Lambda^n_A(E, F)$ and $\Lambda^n_{A_U}(E, F)$.

We verify that $\sigma_n \lambda_U = (\sigma_n \lambda)_U$, $\sigma \in \Sigma_n$. It is enough to check this property for σ a transposition of neighbouring elements, since Σ_n is generated by these transpositions. Let σ be the transposition $\{i \leftrightarrow i+1\}$. With the above shorthands we have $\chi(\xi, \sigma) = (-1)^{\alpha_i \alpha_{i+1}}$ and $\chi(u \otimes \xi, \sigma) = (-1)^{(s_i+\alpha_i)(s_{i+1}+\alpha_{i+1})}$. Now, from (3.9) we have

$$\begin{aligned}
&(\sigma_i \lambda_U)(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)\\
&= (-1)^{(s_i+\alpha_i)(s_{i+1}+\alpha_{i+1})} \lambda_U(u_1 \otimes \xi_1, \ldots, u_{i-1} \otimes \xi_{i-1}, u_{i+1} \otimes \xi_{i+1}, u_i \otimes \xi_i, \ldots, u_n \otimes \xi_n)\\
&= (-1)^{\alpha_i \alpha_{i+1} + s_i s_{i+1}} \varepsilon u_1 \ldots u_{i-1} u_{i+1} u_i u_{i+2} \ldots u_n \otimes \lambda(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \xi_i, \ldots, \xi_n)\\
&\doteq \varepsilon u_1 \ldots u_n \otimes (\sigma_i \lambda)(\xi_1, \ldots, \xi_n) = (\sigma_i \lambda)_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n). \qquad (3.11)
\end{aligned}$$

Now given a choice of $\alpha_i \in \mathbf{Z}/2$ for $i = 1, \ldots, n$, assume that the restriction of λ_U to $E_U^{(\alpha_1)} \times \ldots \times E_U^{(\alpha_n)}$ vanish for all U. We may take for U a Grassmann algebra and choose $u_i \in U^{\alpha_i}$, $i = 1, \ldots, n$ with $u_1 u_2 \ldots u_n \neq 0$. The vanishing of the expression (3.9) for all $\xi_i \in E$ then implies the vanishing of λ. A similar argument proves the last statement of the proposition.

3.4. *Remark.* (i) Arguing exactly as before, we would obtain a homomorphism of left A-modules $\lambda \mapsto \lambda_U$ from $\mathscr{L}_A^n(E^{(1)}, \ldots, E^{(n)}, F)$ to $\mathscr{L}_{A_U}^n(E_U^{(1)}, \ldots, E_U^{(n)}, F_U)$ for arbitrary unital graded left A-modules $E^{(1)}, \ldots, E^{(n)}, F$.

(ii) Using the identification of ordered-permuted tensor products of graded vector spaces discussed in Appendix (A.1), (3.9) can be written in the form

$$\lambda_U(u_1 \otimes \ldots \otimes u_n \otimes \xi_1 \otimes \ldots \otimes \xi_n) = (-1)^{\partial\lambda \sum_{i=1}^{n} \partial u_i} u_1 \ldots u_n \otimes \lambda(\xi_1, \ldots, \xi_n). \qquad (3.9a)$$

We now apply the foregoing results to extensions of graded Lie–Cartan pairs by a graded commutative algebra.

3.5. Proposition. *With (L, A) a unital graded Lie–Cartan pair, U a unital graded commutative algebra*[23] *and A_U, L_U as in* (3.1), (3.2), *the pair (L_U, A_U) becomes a unital graded Lie–Cartan pair over U if we define as follows the products of $u \otimes \xi \in L_U$ by $v \otimes a \in A_U$*

$$(u \otimes \xi)(v \otimes a) = (-1)^{\partial\xi \partial v} uv \otimes \xi a, \qquad (3.12)$$

$$(v \otimes a)(u \otimes \xi) = (-1)^{\partial a \partial u} vu \otimes a\xi, \quad u, v \in U, \ \xi \in L^{\cdot}, \ a \in A^{\cdot}. \qquad (3.13)$$

Furthermore, with V a unital graded left A-module and ϱ a V-connection, letting $\varrho(u \otimes \xi)$, $u \otimes \xi \in L_U$ act on V_U as a graded tensor product of endomorphisms:

$$\varrho(u \otimes \xi) = u \otimes \varrho(\xi), \qquad (3.14)$$

[23] Both real or complex.

i.e.

$$\varrho(u \otimes \xi)(v \otimes X) = (-1)^{\partial\xi \partial v} uv \otimes \varrho(\xi) X, \quad \xi \in L^{\cdot}, \ v \in U^{\cdot}, \qquad (3.14a)$$

we define a V_U-connection ϱ, which is local iff the original V-connection is local, and whose curvature Ω is given by

$$\Omega(u \otimes \xi, v \otimes \eta) = (-1)^{\partial\xi \partial v} uv \otimes \Omega(\xi, \eta), \quad u \in U, \ v \in U^{\cdot}, \ \xi \in L^{\cdot}, \ \eta \in L, \qquad (3.15)$$

where $\otimes$ on the r.h.s. denotes a graded tensor product of endomorphisms, i.e. one has, for $w \in U^{\cdot}$, $X \in V$, $\eta \in L^{\cdot}$

$$\Omega(u \otimes \xi, v \otimes \eta)(w \otimes X) = (-1)^{\partial\xi \partial v + \partial w(\partial\xi + \partial\eta)} uvw \otimes \Omega(\xi, \eta) X. \qquad (3.15a)$$

Proof: We check that (L_U, A_U) is a graded Lie–Cartan pair. As noted above, $L_U = U \otimes L$ is a graded Lie algebra whilst $A_U = U \otimes A$ is a unital graded commutative algebra. Let $\xi, \eta \in L^{\cdot}$, $a, b \in A^{\cdot}$, $u, v, w, z \in U$ and $X \in V$ be homogeneous.

Check of property (1.1): we have, using (3.12),

$$\partial((u \otimes \xi)(b \otimes a)) = \partial((b \otimes a)(u \otimes \xi)) = \partial(u \otimes \xi) + \partial(b \otimes a). \qquad (3.16)$$

Check of property (1.3): we have, using (1.3), (3.12),

$$\begin{aligned}
(w \otimes \xi)\{(u \otimes a)(v \otimes b)\} &= (-1)^{\partial a \partial v}(w \otimes \xi)(uv \otimes ab)\\
&= (-1)^{\partial a \partial v + \partial\xi(\partial u + \partial v)} wuv \otimes \{(\xi a) b + (-1)^{\partial\xi \partial a} a(\xi b)\}\\
&= (-1)^{\partial a \partial v + \partial\xi(\partial u + \partial v)} wuv \otimes (\xi a) b +\\
&\quad + (-1)^{\partial a(\partial v + \partial\xi) + \partial\xi(\partial u + \partial v) + \partial w \partial u} uwv \otimes a(\xi b)\\
&= (-1)^{\partial\xi \partial u}(wu \otimes \xi a)(v \otimes b) +\\
&\quad + (-1)^{\partial a(\partial w + \partial\xi) + \partial\xi(\partial u + \partial v) + \partial w \partial u}(u \otimes a)(wv + \xi b)\\
&= \{(w \otimes \xi)(u \otimes a)\}(v \otimes b) +\\
&\quad + (-1)^{(\partial w + \partial\xi)(\partial a + \partial u)}(u \otimes a)\{(w \otimes \xi)(v \otimes b)\}. \qquad (3.17)
\end{aligned}$$

Check of property (1.4): we have, using (A.8), (1.1), (1.4) and denoting with a dot the product from the left

$$\begin{aligned}
[(u \otimes \xi)^{\cdot}, (v \otimes \eta)^{\cdot}](w \otimes X) &= (-1)^{\partial\xi \partial v}\{(uv^{\cdot}) \otimes ([\xi, \eta]^{\cdot})\}(w \otimes X)\\
&= (-1)^{\partial\xi \partial v + \partial w(\partial\xi + \partial\eta)} uvw \otimes [\xi, \eta] X\\
&= (-1)^{\partial\xi \partial v}\{uv \otimes [\xi, \eta]\}(w \otimes X)\\
&= [u \otimes \xi, v \otimes \eta](w \otimes X). \qquad (3.18)
\end{aligned}$$

Check of property (1.5): one has

$$(u\otimes a)\,\{(v\otimes b)(w\otimes\xi)\} = \{(u\otimes a)(v\otimes b)\}\,(w\otimes\xi) \tag{3.19}$$

by (A.7).

Check of property (1.6): one has, by (1.6)

$$\mathit{1}\otimes(z\otimes\xi) = (\mathit{1}\otimes\mathit{1})(z\otimes\xi). \tag{3.20}$$

Check of property (1.7): one has, using (3.4), (1.7), (3.8), (3.9)

$$\begin{aligned}(u\otimes a)\,\{(w\otimes\xi)(v\otimes b)\} &= (-1)^{\partial\xi\partial v}(u\otimes a)(wv\otimes\xi b)\\ &= (-1)^{\partial\xi\partial v+\partial a(\partial w+\partial v)}\,uwv\otimes a\xi b\\ &= (-1)^{\partial a\partial w}(uw\otimes a\xi)(v\otimes b)\\ &= \{(u\otimes a)(w\otimes\xi)\}\,(v\otimes b).\end{aligned} \tag{3.21}$$

Check of property (1.8): one has, using (1.8), (3.4)

$$\begin{aligned}[w\otimes\xi,\,(u\otimes a)(z\otimes\eta)] &= (-1)^{\partial a\partial z}[w\otimes\xi,\,uz\otimes a\eta]\\ &= (-1)^{\partial a\partial z+\partial\xi(\partial u+\partial z)}\,wuz\otimes\{(-1)^{\partial\xi\partial a}a[\xi,\eta]+(\xi a)\eta\}\\ &= (-1)^{\partial a\partial z+\partial\xi(\partial u+\partial z+\partial a)+\partial w\partial u}\,uwz\otimes a[\xi,\eta]+\\ &\quad+(-1)^{\partial a\partial z+\partial\xi(\partial u+\partial z)}\,wuz\otimes(\xi a)\eta\\ &= (-1)^{\partial\xi(\partial u+\partial z+\partial a)+\partial w(\partial u+\partial a)}(u\otimes a)(wz\otimes[\xi,\eta])+\\ &\quad+(-1)^{\partial\xi\partial u}(wu\otimes\xi a)(z\otimes\eta)\\ &= (-1)^{(\partial u+\partial a)(\partial w+\partial\xi)}(u\otimes a)\,[w\otimes\xi,\,z\otimes\eta]+\\ &\quad+\{(w\otimes\xi)(u\otimes a)\}\,(z\otimes\eta).\end{aligned} \tag{3.22}$$

To check that (L_U, A_U) is a graded Lie–Cartan pair over U we must show that $\hat{\xi}u = 0$ for $\hat{\xi}\in L_U$, $u\in U\subset A_U$. This follows from

$$(v\otimes\xi)(u\otimes\mathit{1}) = (-1)^{\partial\xi\partial u}\,vu\otimes\xi\mathit{1} = 0$$

by (3.12) and footnote 9 in Section 1.

We now check that ϱ specified by (3.14) is a V_U-connection. The two first properties (1.9) are obvious, we check the last one: we have, from (3.15), (A.8)

$$\begin{aligned}\varrho(u\otimes\xi)(v\otimes a) &= \{u\otimes\varrho(\xi)\}\,(v\otimes a)\\ &= (-1)^{\partial\xi\partial v}\,uv\otimes\varrho(\xi)\,a\\ &= (-1)^{(\partial\xi+\partial u)\partial v+\partial\xi\partial a}\,vu\otimes a\varrho(\xi)+(-1)^{\partial\xi\partial v}\,uv\otimes\xi a\\ &= (-1)^{(\partial\xi+\partial u)(\partial v+\partial a)}\,\{(v\otimes a)\big(u\otimes\varrho(\xi)\big)+(u\otimes\xi)(v\otimes a)\}.\end{aligned} \tag{3.23}$$

Finally we check (3.15): we have, using (A.8) from (3.13)

$$\begin{aligned}&[\varrho(u\otimes\xi),\,\varrho(v\otimes\eta)]-\varrho([u\otimes\xi,\,v\otimes\eta])\\ &\qquad= [u\otimes\varrho(\xi),\,v\otimes\varrho(\eta)]-(-1)^{\partial\xi\partial v}\varrho(uv\otimes[\xi,\eta])\\ &\qquad= (-1)^{\partial\xi\partial v}\,uv\otimes\{[\varrho(\xi),\varrho(\eta)]-\varrho([\xi,\eta])\}.\end{aligned} \tag{3.24}$$

We now show how the map $\lambda\mapsto\lambda_U$ applied to $\Lambda^*_A(L, V)$ intertwines the classical operators of Section 2.

3.6. PROPOSITION. *Let (L, V) be a unital graded Lie–Cartan pair, with ϱ a V-connection of (L, A), and δ_0, $\varrho\wedge$, δ_ϱ, $\theta_0(\xi)$, $\varrho(\xi)$, $\theta_\varrho(\xi)$, $i(\xi)$, $a\cdot$, $\delta a\wedge$, $\Omega\wedge$ the corresponding classical operators acting on $\Lambda^*(L, V)$.*

Given a graded-commutative algebra U, we consider the corresponding graded Lie–Cartan pair (L_U, A_U) and V_U-connection ϱ (cf. Proposition 3.5) and the classical operators attached to the latter acting on $\Lambda^*(L_U, V_U)$. With $\lambda\mapsto\lambda_U$ the map of Proposition 3.3 for $E = L$ and $F = V$ we then have the following intertwining properties: for $\lambda\in\Lambda^*(L, V)$, ξ, $\eta\in L$, $u\in U$, $a\in A$

$$\delta_0(\lambda_U) = (\delta_0\lambda)_U, \tag{3.25}$$

$$\varrho\wedge(\lambda_U) = (\varrho\wedge\lambda)_U, \tag{3.26}$$

$$\delta_\varrho(\lambda_U) = (\delta_\varrho\lambda)_U, \tag{3.27}$$

$$\theta_0(u\otimes\xi)(\lambda_U) = (u\otimes id)\,\{\theta_0(\xi)\lambda\}_U, \tag{3.28}$$

$$\varrho(u\otimes\xi)(\lambda_U) = (u\otimes id)\,\{\varrho(\xi)\lambda\}_U, \tag{3.29}$$

$$\theta_\varrho(u\otimes\xi)(\lambda_U) = (u\otimes id)\,\{\theta_\varrho(\xi)\lambda\}_U, \tag{3.30}$$

$$i(u\otimes\xi)(\lambda_U) = (u\otimes id)\,\{i(\xi)\lambda\}_U, \tag{3.31}$$

$$(u\otimes a)\cdot(\lambda_U) = (u\otimes id)(a\cdot\lambda)_U, \tag{3.32}$$

$$\delta(u\otimes a)\wedge\lambda_U = (u\otimes id)(\delta a\wedge\lambda)_U, \tag{3.33}$$

$$\Omega\wedge(\lambda_U) = (\Omega\wedge\lambda)_U, \tag{3.34}$$

$$\Omega(u\otimes\xi,\,v\otimes\eta)(\lambda_U) = (-1)^{\partial\xi\partial v}(uv\otimes id)\,\{\Omega(\xi,\eta)\lambda\}_U. \tag{3.35}$$

Proof: Let $u_i\in U^{\cdot}$, $\xi_i\in L^{\cdot}$, $i = 1, 2, \ldots, n+1$.

Check of (3.25): we have, since $[u_1\otimes\xi_1,\,u_2\otimes\xi_2] = (-1)^{\partial\xi_1\partial u_2}u_2u_1\otimes[\xi_1,\xi_2]$ (cf. (A.28)), using (3.9) and the equalities $\partial\lambda = \partial\lambda_U = \partial\lambda^\delta$

$$\begin{aligned}&(\lambda_U)^\delta(u_1\otimes\xi_1,\ldots,u_{n+1}\otimes\xi_{n+1})\\ &\qquad= (-1)^{\partial\xi_1\partial u_1}\lambda_U(u_1u_2\otimes[\xi_1,\xi_2],\,u_3\otimes\xi_3,\ldots,u_{n+1}\otimes\xi_{n+1})\end{aligned}$$

$$= (-1)^{\partial\lambda \sum_{i=1}^{n} \partial u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j} u_1 u_2 \ldots u_{n+1} \otimes \lambda^\delta(\xi_1, \ldots, \xi_{n+1})$$

$$= (\lambda^\delta)_U (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1}). \tag{3.36}$$

Hence $(\lambda_u)^\delta = (\lambda^\delta)_u$. By Proposition 3.3 if follows that

$$-\frac{2}{n(n+1)} \delta_0 (\lambda_U) = A_{n+1} \{(\lambda_U)^\delta\} = A_{n+1} \{(\lambda^\delta)_U\}$$

$$= \{A_{n+1}(\lambda^\delta)\}_U = -\frac{2}{n(n+1)} (\delta_0 \lambda)_U. \tag{3.37}$$

Check of (3.26): we have now

$$(\lambda_U)^\varrho (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1})$$

$$= (-1)^{\partial\lambda(\partial u_1 + \partial\xi_1)} \varrho(u_1 \otimes \xi_1) \lambda_U (u_2 \otimes \xi_2, \ldots, u_{n+1} \otimes \xi_{n+1})$$

$$= (-1)^{\partial\lambda(\partial u_1 + \partial\xi_1) + \partial\lambda \sum_{i=2}^{n+1} u_i + \sum_{2 \leqslant i < j \leqslant n+1} \partial\xi_i \partial u_j} \times$$

$$\times \varrho(u_1 \otimes \xi_1) \{u_2 \otimes \ldots \otimes u_{n+1} \otimes \lambda(\xi_2, \ldots, \xi_{n+1})$$

$$= (-1)^{\partial\lambda\left(\partial\xi_1 + \sum_{i=1}^{n+1} \partial u_i\right) + \sum_{2 \leqslant i < j \leqslant n+1} \partial\xi_i \partial u_j + \partial\xi_1 (\partial u_2 + \ldots + \partial u_{n+1})} \times$$

$$\times u_1 u_2 \ldots u_{n+1} \otimes \varrho(\xi_1) \lambda(\xi_2, \ldots, \xi_{n+1})$$

$$= (-1)^{\partial\lambda \sum_{i=1}^{n+1} \partial u_i + \sum_{1 \leqslant i < j \leqslant n+1} \partial\xi_i \partial u_j} u_1 u_2 \ldots u_{n+1} \otimes \lambda^\varrho (\xi_1, \ldots, \xi_{n+1})$$

$$= (\lambda^\varrho)_U (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1}), \tag{3.38}$$

hence $(\lambda_U)^\varrho = (\lambda^\varrho)_U$, and

$$\frac{1}{n+1} \varrho \wedge \lambda_U = A_{n+1} \{(\lambda_U)^\varrho\} = A_{n+1} \{(\lambda^\varrho)_U\}$$

$$= \{A_{n+1}(\lambda^\varrho)\}_U = \frac{1}{n+1} (\varrho \wedge \lambda)_U. \tag{3.39}$$

(3.27) then follows from (3.25), (3.26).

Check of (3.28): since $[u \otimes \xi, u_1 \otimes \xi_1] = (-1)^{\partial\xi \partial u_1} u u_1 \otimes [\xi, \xi_1]$ we have

$$(\lambda_U)(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)$$

$$= (-1)^{\partial\xi \partial u_1 + \partial\lambda(\partial u_1 + \partial\xi)} \lambda(u u_1 \otimes [\xi, \xi_1], u_2 \otimes \xi_2, \ldots, u_n \otimes \xi_n)$$

$$= (-1)^{\partial\lambda\left(\partial\xi + \sum_{i=1}^{n} \partial u_i\right) + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j + \partial\xi \sum_{i=1}^{n} \partial u_i} \times$$

$$\times u u_1 \ldots u_n \otimes \lambda([\xi, \xi_1], \xi_2, \ldots, \xi_n)$$

$$= (-1)^{(\partial\lambda + \partial\xi) \sum_{i=1}^{n} \partial u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j} (u \otimes id) \{u_1 \ldots u_n \otimes \lambda^\xi (\xi_1, \ldots, \xi_n)\}$$

$$= (u \otimes id) \{\lambda^\xi\}_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n). \tag{3.40}$$

Hence

$$(\lambda_U)^{u \otimes \xi} = (u \otimes id)(\lambda^\xi)_U - \frac{1}{n} \theta_0 (u \otimes \xi)(\lambda_U) = A_n (\lambda_U)^{u \otimes \xi} = (u \otimes id) A_n \{\lambda^\xi\}_U$$

$$= (u \otimes id) \{A_n \lambda^\xi\}_U = -\frac{1}{n} (u \otimes id) \{\theta_0(\xi) \lambda\}_U. \tag{3.41}$$

Check of (3.29): by (3.9) and (3.14a), we have

$$\{\varrho(u \otimes \xi)(\lambda_U)\}(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)$$

$$= \varrho(u \otimes \xi) \{\lambda_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)\}$$

$$= (-1)^{\partial\lambda \sum_{i=1}^{n} u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j} \varrho(u \otimes \xi) \{u_1 \ldots u_n \otimes \lambda(\xi_1, \ldots, \xi_n)\}$$

$$= (-1)^{(\partial\lambda + \partial\xi) \sum_{i=1}^{n} \partial u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j} u u_1 \ldots u_n \otimes \{\varrho(\xi) \lambda\}(\xi_1, \ldots, \xi_n)$$

$$= (u \otimes id) (\varrho(\xi) \lambda)_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n). \tag{3.42}$$

Check of (3.31): one has

$$\{i(u \otimes \xi)(\lambda_U)\}(u_1 \otimes \xi_1, \ldots, u_{n-1} \otimes \xi_{n-1})$$

$$= (-1)^{\partial\lambda(\partial u + \partial\xi)} \lambda_U (u \otimes \xi, u_1 \otimes \xi_1, \ldots, u_{n-1} \otimes \xi_{n-1})$$

$$= (-1)^{\partial\lambda(\partial u + \partial\xi) + \partial\lambda(\partial u + \sum_{i=1}^{n-1} \partial u_i) + \sum_{1 \leqslant i < j \leqslant n-1} \partial\xi_i \partial u_j + \partial\xi \sum_{i=1}^{n-1} \partial u_i} \times$$

$$\times u u_1 \ldots u_{n-1} \otimes \lambda(\xi, \xi_1, \ldots, \xi_{n-1})$$

$$= (-1)^{(\partial\lambda + \partial\xi) \sum_{i=1}^{n-1} \partial u_i + \sum_{1 \leqslant i < j \leqslant n-1} \partial\xi_i \partial u_j} (u \otimes id) \{u_1 \ldots u_n \otimes \{i(\xi)\lambda\}(\xi_1, \ldots, \xi_{n-1})$$

$$= (u \otimes id) \{i(\xi)\lambda\}_U (u_1 \otimes \xi_1, \ldots, u_{n-1} \otimes \xi_{n-1}). \qquad (3.43)$$

Check of (3.32): one has

$$\{(u \otimes a)(\lambda_U)\}(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)$$

$$= (u \otimes a) \lambda_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)$$

$$= (-1)^{\partial\lambda \sum_{i=1}^{n} \partial u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j + \partial a \sum_{i=1}^{n} \partial u_i} \times$$

$$\times u u_1 \ldots u_n \otimes a\lambda(\xi_1, \ldots, \xi_n)$$

$$= (u \otimes id)(a\lambda)_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n). \qquad (3.44)$$

Check of (3.33): one has

$$(\lambda_U)^{u \otimes a}(u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1})$$

$$= (-1)^{(\partial u_1 + \partial\xi_1)(\partial u + \partial a + \partial\lambda)} \{(u_1 \otimes \xi_1)(u \otimes a)\} \lambda_U (u_2 \otimes \xi_2, \ldots, u_{n+1} \otimes \xi_{n+1})$$

$$= (-1)^{(\partial u_1 + \partial\xi_1)(\partial u + \partial a + \partial\lambda) + \partial\xi_1 \partial + \partial\lambda \sum_{i=2}^{n+1} \partial u_i + \sum_{2 \leqslant i < j \leqslant n+1} \partial\xi_i \partial u_j} \times$$

$$\times (u_1 u \otimes \xi_1 a)(u_2 \ldots u_{n+1} \otimes \lambda(\xi_2, \ldots, \xi_{n+1}))$$

$$= (-1)^{(\partial u_1 + \partial\xi_1)(\partial a + \partial\lambda) + \partial\lambda \sum_{i=2}^{n+1} \partial u_i + \sum_{2 \leqslant i < j \leqslant n+1} \partial\xi_i \partial u_j + (\partial\xi_1 + \partial a) \sum_{i=2}^{n+1} \partial u_i} \times$$

$$\times u u_1 \ldots u_{n+1} \otimes (\xi_1 a) \lambda(\xi_2, \ldots, \xi_{n+1})$$

$$= (-1)^{(\partial\lambda + \partial a) \sum_{i=1}^{n+1} \partial u_i + \sum_{1 \leqslant i < j \leqslant n+1} \partial\xi_i \partial u_j} \times$$

$$\times u u_1 \ldots u_{n+1} \otimes \lambda^a(\xi_1, \ldots, \xi_{n+1})$$

$$= (u \otimes id)(\lambda^a)_U (u_1 \otimes \xi_1, \ldots, u_{n+1} \otimes \xi_{n+1}), \qquad (3.45)$$

hence $(\lambda_U) = (u \otimes id)(\lambda^a)_U$. Then

$$\frac{1}{n+1} \delta(u \otimes a) \wedge (\lambda_U) = A_{n+1} (\lambda_U)^{u \otimes a} = (u \otimes id) A_{n+1} \{(\lambda^a)_U\}$$

$$= (u \otimes id) \{A_{n+1}(\lambda^a)\}_U = \frac{1}{n+1} (u \otimes id)(\delta a \wedge \lambda)_U. \qquad (3.46)$$

Check of (3.34): from (3.9), (3.15a), we have

$$\{\Omega \wedge (\lambda_U)\}(u_1 \otimes \xi_1, \ldots, u_{n+2} \otimes \xi_{n+2})$$

$$= (-1)^{\partial\lambda(\partial u_1 + \partial u_2 + \partial\xi_1 + \partial\xi_2)} \Omega(u_1 \otimes \xi_1, u_2 \otimes \xi_2) \lambda_U (u_3 \otimes \xi_3, \ldots, u_{n+2} \otimes \xi_{n+2})$$

$$= (-1)^{\partial\lambda(\partial u_1 + \partial u_2 \partial\xi_1 + \partial\xi_2) + \partial\lambda \sum_{i=2}^{n+2} \partial u_i + \sum_{3 \leqslant i < j \leqslant n+2} \partial\xi_i \partial u_j} \times$$

$$\times \Omega(u_1 \otimes \xi_1, u_2 \otimes \xi_2) \{u_3 \ldots u_{n+2} \otimes \lambda(\xi_3, \ldots, \xi_{n+2})\}$$

$$= (-1)^{\partial\lambda \sum_{i=1}^{n+2} \partial u_i + \sum_{3 \leqslant i < j \leqslant n+2} \partial\xi_i \partial u_j + \partial\xi_1 \partial u_2 + (\partial\xi_1 + \partial\xi_2) \sum_{i=3}^{n+2} \partial u_i} \times$$

$$\times u_1 \ldots u_{n+2} \otimes (\Omega \wedge \lambda)(\xi_1, \ldots, \xi_{n+2})$$

$$= (\Omega \wedge \lambda)_U (u_1 \otimes \xi_1, \ldots, u_{n+2} \otimes \xi_{n+2}). \qquad (3.47)$$

Check of (3.35): from (2.5), (3.15), we have

$$\{\Omega(u \otimes \xi, v \otimes \eta)(\lambda_U)\}(u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n)$$

$$= (-1)^{\partial\xi \partial v + \partial\lambda \sum_{i=1}^{n} \partial u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j} \times$$

$$\times \{uv \otimes \Omega(\xi, \eta)\} \{u_1, \ldots, u_n \otimes \lambda(\xi_1, \ldots, \xi_n)\} \times$$

$$\times (-1)^{\partial\xi \partial v + (\partial\lambda + \partial\xi + \partial\eta) \sum_{i=1}^{n} \partial u_i + \sum_{1 \leqslant i < j \leqslant n} \partial\xi_i \partial u_j} \times$$

$$\times uvu_1 \ldots u_n \otimes \Omega(\xi, \eta) \lambda(\xi_1, \ldots, \xi_n)$$

$$= (uv \otimes id) \{\Omega(\xi, \eta)\lambda\}_U (u_1 \otimes \xi_1, \ldots, u_n \otimes \xi_n). \qquad (3.48)$$

3.7. *Remark*. Restricting the above classical operators to the arguments from the even part L_U^0, we thus arrive at the classical (Abelian) situation discussed in [1].

4. Derivation properties

In this section we describe two types of derivations properties fulfilled by the classical operators attached to a V-connection (cf. Section 2). On the one hand, if V carries a bilinear[24] (not necessarily associative) product, this product extends as a graded wedge product of $\Lambda^*(L, V)$ which becomes an algebra[25] for which the classical operators are derivations. On the other hand, for a general A-module V (which we then prefer to consider as a right A-module, cf. (A.40) in Appendix A), $\Lambda^*(L, V)$ is a right $\Lambda^*(L, A)$-module and the classical operators are derivations of this module. Both are particular cases of a more general situation when there is given a bilinear map $V\times V'\to V''$ (see Proposition 4.4 below).

4.1. DEFINITION. Let (L, A) be a graded Lie–Cartan pair, with V, V', V'' unital left A-modules and let there be given a graded bilinear product

$$V^i\times V'^j\ni(X, X')\to XX'\in V''^{i+j},\quad i, j\in \mathbf{Z}/2. \tag{4.1}$$

Given $\lambda\in\mathscr{L}^n(L, V)^\cdot$, $\mu\in\mathscr{L}^n(L, V')^\cdot$, their *graded tensor product* $\lambda\otimes\mu\in\mathscr{L}^{n+m}(L, V'')$ is defined by

$$\begin{aligned}\lambda\otimes\mu(\xi_1, \ldots, \xi_{n+m}) &= (-1)^{\partial\mu\sum_{i=1}^n\partial\xi_i}\lambda(\xi_1, \ldots, \xi_n)\,\mu(\xi_{n+1}, \ldots, \xi_{n+m}),\\ &\xi_1, \ldots, \xi_{n+m}\in L^\cdot,\end{aligned} \tag{4.2}$$

their *graded wedge product* $\lambda\wedge\mu\in\Lambda(L, V'')$ being then given by

$$\lambda\wedge\mu = ((n+m)!/n!\,m!)\,A_{n+m}(\lambda\otimes\mu), \tag{4.3}$$

where A_n denotes the graded antisymmetrizer determined by the graded alternate character χ (cf. Definition B.3).

4.2. *Remark*. The elements of

$$V\otimes_{\mathbf{R}}\mathscr{L}^n(L, \mathbf{R})(V\otimes_{\mathbf{C}}\mathscr{L}^n(L, \mathbf{C}),$$

resp. $V\otimes_A\mathscr{L}^n_A(L, A)$ can be considered as V-valued $\mathbf{R}$-n-linear ($\mathbf{C}$-n-linear) forms, on L, resp. V-valued n-A-linear forms on L, the identification being in both cases given by

$$\begin{aligned}(X\otimes\varphi)(\xi_1, \ldots, \xi_n) &= (-1)^{\partial X(\partial\varphi+\sum_{i=1}^n\partial\xi_i)}\varphi(\xi_1, \ldots, \xi_n)X\\ &= X\varphi(\xi_1, \ldots, \xi_n),\quad \text{cf. (A.42)},\end{aligned} \tag{4.4}$$

$$X\otimes\varphi\in V\otimes_{\mathbf{R}}\mathscr{L}^n(L, \mathbf{R})(V\otimes_{\mathbf{C}}\mathscr{L}^n(L, \mathbf{C})),\quad \text{resp. } V\otimes_A\mathscr{L}^n_A(L, A).$$

[24] $\mathbf{R}(\mathbf{C})$-bilinear or A-bilinear. In the latter case, and if the connection is local, one may consider $\Lambda^*_A(L, V)$.

In terms of these descriptions the products (4.2) and (4.3) are given as follows: for $X\in V$, $X'\in V'$, φ, $\varphi'\in\mathscr{L}^*(L, \mathbf{R})$ $(\in\mathscr{L}^*(L, \mathbf{C}))$ we have

$$(X\otimes\varphi)\otimes(X'\otimes\varphi') = (-1)^{\partial\varphi\partial X'}(X\cdot X')\otimes(\varphi\otimes\varphi'), \tag{4.5}$$

$$(X\otimes\varphi)\wedge(X'\otimes\varphi') = (-1)^{\partial\varphi\partial X'}(X\cdot X')\otimes(\varphi\wedge\varphi'), \tag{4.6}$$

these formulae holding also for φ, $\varphi'\in\mathscr{L}^*(L, A)$ if one assumes that the map $V\times V'\to V''$ is A-linear

$$\begin{aligned}(aX)\cdot X' = a(X\cdot X'),\quad X\cdot(aX') &= (-1)^{\partial a\partial X}a(X\cdot X'),\\ a\in A,\ X\in V,\ X'&\in V'.\end{aligned} \tag{4.7}$$

(In (4.6) the wedge product is in all cases the one defined in terms of the graded tensor product (B.34).)

Proof: Let $\xi_1, \ldots, \xi_{n+m}\in L^\cdot$. We have, for φ, $\varphi'\in\mathscr{L}^*(L, \mathbf{R})$

$$\begin{aligned}&\{(X\otimes\varphi)\otimes(X'\otimes\varphi')\}(\xi_1, \ldots, \xi_{n+m})\\ &= (-1)^{(\partial X'+\partial\varphi')\sum_{i=1}^n\partial\xi_i}\{X\varphi(\xi_1, \ldots, \xi_n)\}\cdot\{X'\varphi'(\xi_{n+1}, \ldots, \xi_{n+m})\}\\ &= (-1)^{\partial X'\sum_{i=1}^n\partial\xi_i}(X\cdot X')\{\varphi\otimes\varphi')(\xi_1, \ldots, \xi_{n+m})\},\end{aligned} \tag{4.8a}$$

where $\sum_{i=1}^n\partial\xi_i$ may be replaced by $\partial\varphi$ since φ has values in $\mathbf{R}$ (in $\mathbf{C}$). For φ, $\varphi'\in\mathscr{L}^*(L, A)$ the above calculation holds for the two first lines, the sequel being

$$\begin{aligned}&= (-1)^{\partial\varphi'\sum_{i=1}^n\partial\xi_i+\partial X'\partial\varphi}(X\cdot X')\,\varphi(\xi_1, \ldots, \xi_n)\,\varphi'(\xi_{n+1}, \ldots, \xi_{n+m})\\ &= (-1)^{\partial X'\partial\varphi}(X\cdot X')(\varphi\otimes\varphi')(\xi_1, \ldots, \xi_{n+m})\\ &= (-1)^{\partial X'}\{(X\cdot X')\otimes(\varphi\otimes\varphi')\}(\xi_1, \ldots, \xi_{n+m}).\end{aligned} \tag{4.8b}$$

We checked (4.5): (4.6) then follows from (4.3) due to the fact that

$$A_{n+m} = A_n\otimes A_m\quad\text{and}\quad A_n(X\otimes\varphi) = X\otimes A_n\varphi.$$

4.3. PROPOSITION. *With the notations and definitions in Definition* 4.1 *we have that*

(i) *If* λ *and* λ' *belong to* $\mathscr{L}^*_A(L, V)$, *and* $\mathscr{L}^*_A(L, V)$ *resp., then* $\lambda\otimes\lambda'$ *and* $\lambda\wedge\lambda'$ *are in* $\mathscr{L}^*_A(L, V'')$.

and $\wedge$ *defined in* (4.2), *resp.* (4.3) $(\mathscr{L}^*(L, V)\ \otimes)$ *and* $(\Lambda^*(L, V, \wedge)$ *are then associative algebras with respective subalgebras* $\mathscr{L}^*_A(L, V)$ *and* $\Lambda^*_A(L, V)$.

(iii) *If* $V = V' = V''$ *and the product* $\cdot$ *of* V *is graded commutative, the graded wedge product of* $\Lambda^*(L, V)$ *fulfils*

$$\mu\wedge\lambda = (-1)^{n+m+\partial\mu\partial\lambda}\lambda\wedge\mu,\qquad \lambda\in\Lambda^n(L, V)^{\cdot},\quad \mu\in\Lambda^m(L, V)^{\cdot}. \tag{4.9}$$

Proof: (i) The proof is the same as that of the A-linearity of $\lambda\otimes\mu$ in Proposition B.6, cf. equations (B.38), (B.39) in Appendix B.

(ii) We have, for $X, X', X''\in V^{\cdot}$, $\varphi, \varphi', \varphi''\in\mathscr{L}^*(L, \boldsymbol{R})^{\cdot}(\mathscr{L}^*(L, \boldsymbol{C})^{\cdot})$,

$$\begin{aligned}&\{(X\otimes\varphi)\wedge(X'\otimes\varphi')\}\wedge(X''\otimes\varphi'')\\ &\quad= (-1)^{\partial\varphi\partial X'}\{(X\cdot X')\cdot X''\}\otimes(\varphi\wedge\varphi'\wedge\varphi'')\\ &\quad= (-1)^{\partial\varphi\partial X'+\partial X'(\partial\varphi+\partial\varphi')}\{(X\cdot X')\cdot X''\}\otimes(\varphi\wedge\varphi'\wedge\varphi''),\end{aligned} \tag{4.10}$$

whilst

$$\begin{aligned}(X\otimes\varphi)\{(X'\otimes\varphi')\wedge(X''\otimes\varphi'')\} &= (-1)^{\partial\varphi\partial X''}(X\otimes\varphi)\{(X\cdot X')\otimes(\varphi'\wedge\varphi'')\}\\ &= (-1)^{\partial\varphi'\partial X''+\partial\varphi(\partial X'+\partial X'')}\{X\cdot(X'\cdot X'')\}\otimes(\varphi\wedge\varphi'\wedge\varphi'')\end{aligned} \tag{4.10a}$$

showing that the product $\wedge$ is associative if this holds for the product. The proof for the tensor product $\otimes$ is identical (or can be checked directly as in (B.40)).

(iii) One has then, for $X, X'\in V$, $\varphi, \varphi'\in\Lambda^*(L, V)$

$$\begin{aligned}(X'\otimes\varphi')\wedge(X\otimes\varphi) &= (-1)^{\partial\varphi'\partial X}(X'\cdot X)\otimes(\varphi'\wedge\varphi)\\ &= (-1)^{\partial\varphi'\partial X+\partial X'\partial X+\partial\varphi'\partial\varphi}(X\cdot X')\otimes(\varphi\wedge\varphi')\\ &= (-1)^{\partial(X\otimes\varphi)\partial(X'\otimes\varphi')}(X\otimes\varphi)\wedge(X'\otimes\varphi').\end{aligned} \tag{4.11}$$

4.4. Proposition. *With the graded Lie–Cartan pair* (L, A) *and the* A-*modules* V_i $(i = 1, 2, 3)$ *as in Definition* 4.1, *and with* ϱ_i *being* V_i-*connections, we consider the corresponding classical operators (acting on* $\mathscr{L}^*(L, V_i) = \underset{n\in N}{\otimes}\mathscr{L}^n(L, V_i)$ *as specified in Definition* 2.1 *along with their restrictions to* $\Lambda^n(L, V_i)$, *resp., for* ϱ_i *local, to* $\Lambda^*_A(L, V_i)$ *(cf. Proposition* 2.2). *These operators fulfil the following derivation properties: one has for* $\xi\in L^{\cdot}$, $\lambda_i\in\Lambda^n(L, V_i)^{\cdot}$, $i = 1, 2$

$$i(\xi)(\lambda_1\wedge\lambda_2) = \{i(\xi)\lambda_1\}\wedge\lambda_2+(-1)^{n+\partial\xi\partial\lambda}\lambda_1\wedge i(\xi)\lambda_2 \tag{4.12}$$

and, if ϱ_3 *is the "tensor product" of* ϱ_1 *and* ϱ_2, *i.e. if*

$$\varrho_3(\xi)(X_1\cdot X_2) = \{\varrho_1(\xi)X_1\}\cdot X_2+(-1)^{\partial\xi\partial X_1}X_1\cdot\varrho_2(\xi)X_2,\qquad X_1\in V_1^{\cdot},\ X_2\in V_2^{\cdot}, \tag{4.13}$$ [26]

[26] The results (4.12), (4.17), (4.19), (4.21) hold without the assumption (4.13).

then one has

$$\theta_{\varrho_3}(\xi)(\lambda_1\wedge\lambda_2) = \{\theta_{\varrho_1}(\xi)\lambda_1\}\wedge\lambda_2+(-1)^{\partial\xi\partial\lambda}\lambda_1\wedge\theta_{\varrho_2}(\xi)\lambda_2, \tag{4.14}$$

$$\delta_{\varrho_3}(\lambda_1\wedge\lambda_2) = (\delta_{\varrho_1}\lambda_1)\wedge\lambda_2+(-1)^n\lambda_1\wedge\delta_{\varrho_2}\lambda_2, \tag{4.15}$$

$$\Omega_3\wedge(\lambda_1\wedge\lambda_2) = (\Omega_1\wedge\lambda_1)\wedge\lambda_2+\lambda_1\wedge(\Omega_2\wedge\lambda_2). \tag{4.16}$$

In fact, one has, separately

$$\theta_0(\xi)(\lambda_1\wedge\lambda_2) = \{\theta_0(\xi)\lambda_1\}\wedge\lambda_2+(-1)^{\partial\xi\partial\lambda_1}\lambda_1\wedge\theta_0(\xi)\lambda_2, \tag{4.17}$$

$$\varrho_3(\xi)(\lambda_1\wedge\lambda_2) = \{\varrho_1(\xi)\lambda_1\}\wedge\lambda_2+(-1)^{\partial\xi\partial\lambda_1}\lambda_1\wedge\varrho_2(\xi)\lambda_2, \tag{4.18}$$

resulting from

$$\theta_0(\xi)(\lambda_1\otimes\lambda_2) = \{\theta_0(\xi)\lambda_1\}\otimes\lambda_2+(-1)^{\partial\xi\partial\lambda_1}\lambda_1\otimes\theta_0(\xi)\lambda_2, \tag{4.19}$$

$$\varrho_3(\xi)(\lambda_1\otimes\lambda_2) = \{\varrho_1(\xi)\lambda_1\}\otimes\lambda_2+(-1)^{\partial\xi\partial\lambda_1}\lambda_1\otimes\varrho_2(\xi)\lambda_2\} \tag{4.20}$$

and

$$\delta_0(\lambda_1\wedge\lambda_2) = (\delta_0\lambda_1)\wedge\lambda_2+(-1)^n\lambda_1\wedge\delta_0\lambda_2, \tag{4.21}$$

$$\varrho_3\wedge(\lambda_1\wedge\lambda_2) = (\varrho_1\wedge\lambda_1)\wedge\lambda_2+(-1)^n\lambda_1\wedge(\varrho_2\wedge\lambda_2). \tag{4.22}$$

Proof: Check of (4.12) for $V_i = A$: [27] by linearity, it is enough to check that (4.12) holds for $\lambda = \varphi_1\wedge\ldots\wedge\varphi_m$; $\varphi_1, \ldots, \varphi_m\in\Lambda^1(L, A)$. We operate by induction w.r.t. n. Let $a\in A$, $\varphi\in\Lambda^1(L, A)$, $\mu\in\Lambda^m(L, A)$. We first have, for $\xi_1, \ldots, \xi_n\in L^{\cdot}$,

$$\begin{aligned}i(\xi)(a\wedge\mu)(\xi_1, \ldots, \xi_{m-1}) &= (-1)^{(\partial\xi(\partial a+\partial\mu)}a\mu(\xi, \xi_1, \ldots, \xi_{m-1})\\ &= (-1)^{\partial\xi\partial X}a\{i(\xi)\mu\}(\xi_1, \ldots, \xi_{m-1})\\ &= \{(i(\xi)a\wedge\mu+(-1)^{\partial\xi\partial X}a\wedge i(\xi)\mu\}(\xi_1, \ldots, \xi_{m-1}),\end{aligned} \tag{4.23}$$

further, using (B.43),

$$\begin{aligned}&\{i(\xi)(\varphi\wedge\mu)\}(\xi_1, \ldots, \xi_m)\\ &= (-1)^{\partial\xi(\partial\varphi+\partial\mu)}(\varphi\wedge\mu)(\xi, \xi_1, \ldots, \xi_m)\\ &= (-1)^{\partial\xi(\partial\varphi+\partial\mu)}\{(-1)^{\partial\xi\partial\mu}\varphi(\xi)\mu(\xi_1, \ldots, \xi_m)+\\ &\quad+\sum_{i=1}^m(-1)^{i+\partial\xi_i(\partial\mu+\partial\xi+\sum_{k=1}^{i-1}\partial\xi_k)}\varphi(\xi_i)\mu(\xi, \xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_m)\\ &= (-1)^{\partial\xi\partial\varphi}\varphi(\xi)\mu(\xi_1, \ldots, \xi_m)+(-1)^{\partial\xi(\partial\varphi+\partial\mu)}\sum_{i=1}^m(-1)^{+\partial\xi_i(\partial\mu+\partial\xi+\sum_{k=1}^{i-1}\partial\xi_k)}\times\\ &\quad\times\varphi(\xi_i)\mu(\xi, \xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_m)\\ &= \{\{i(\xi)\varphi\}\wedge\mu-(-1)^{\partial\xi\partial\varphi}\varphi\wedge i(\xi)\mu\}(\xi_1, \ldots, \xi_m).\end{aligned} \tag{4.24}$$

[27] (4.12) can be proved by direct computation. Since this is cumbersome we use tensorial

Assuming now (4.12) to hold for λ and μ, we show that it holds for $\varphi\wedge\lambda$ and μ we have, by what precedes

$$i(\xi)\{\varphi\wedge\lambda\wedge\mu\}=\{i(\xi)\varphi\}\wedge\lambda\wedge\mu-(-1)^{\partial\xi\partial\varphi}\varphi\wedge\{(i(\xi)\lambda)\wedge\mu+(-1)^{n+\partial\xi\partial\lambda}\lambda\wedge i(\xi)\mu\}$$
$$=\{i(\xi)\{\varphi\wedge\lambda\}\}\wedge\mu+(-1)^{n+1+\partial\xi(\partial\varphi+\partial\lambda)}\varphi\wedge\lambda\wedge i(\xi)\mu. \tag{4.25}$$

We checked (4.12) on $\Lambda^n(L, A)$. Now $i(\xi)$ on $\Lambda^n(L, V)=V\otimes_A\Lambda^n(L, A)$ is given by

$$i(\xi)=id\otimes i(\xi) \tag{4.26}$$

in the sense (A.7): indeed, for $X\in V^\cdot$, $\varphi\in\Lambda^n(L, A)$

$$i(\xi)\{X\otimes\varphi\}(\xi_1, \ldots, \xi_{n-1})=(-1)^{\partial\xi(\partial X+\partial\varphi)+\partial X(\partial\varphi+\partial\xi+\sum_{k=1}^{n-1}\partial\xi_k)}\varphi(\xi, \xi_1, \ldots, \xi_{n-1})X$$
$$=(-1)^{\partial\xi\partial X}(X\otimes i(\xi)\varphi)(\xi_1, \ldots, \xi_{n-1}). \tag{4.27}$$

Now, for X, $X'\in V^\cdot$, $\varphi\in\Lambda^n(L, A)$, $\varphi'\in\Lambda^*(L, A)$, we have

$$i(\xi)\{(X\otimes\varphi)\wedge(X'\otimes\varphi')\}=(-1)^{\partial\varphi\partial X'}i(\xi)\{X\cdot X'\otimes(\varphi\wedge\varphi')\}$$
$$=(-1)^{\partial\varphi\partial X'+\partial\xi(\partial X+\partial X')}X\cdot X'\otimes\{(i(\xi)\varphi)\wedge\varphi'+(-1)^{n+\partial\xi\partial\varphi}\varphi\wedge i(\xi)\varphi'\}$$
$$=(-1)^{\partial\varphi\partial X'+\partial\xi(\partial X+\partial X')}\{(-1)^{\partial X'(\partial\xi+\partial\varphi)}(X\otimes i(\xi)\varphi)\wedge(X'\otimes\varphi')+$$
$$+(-1)^{n+\partial\xi\partial\varphi+\partial X'\partial\varphi}(X\otimes\varphi)(X'\otimes i(\xi)\varphi')\}$$
$$=\{i(\xi)(X\otimes\varphi)\}\wedge(X'\otimes\varphi')+(-1)^{n+\partial\xi(\partial X+\partial\varphi)}(X\otimes\varphi)\wedge i(\xi)(X'\otimes\varphi'). \tag{4.28}$$

We proved (4.12).

We now check (4.19) directly: for $\xi_1, \ldots, \xi_{m+n}\in L^\cdot$, we have

$$-\theta_0(\xi)\{\lambda\otimes\mu\}(\xi_1, \ldots, \xi_{m+n})$$
$$=\sum_{i=1}^{n}(-1)^{\partial\xi(\partial\lambda+\partial\mu+\sum_{k=1}^{i-1}\partial\xi_k)+\partial\mu(\partial\xi+\sum_{k=1}^{n}\partial\xi_k)}\lambda(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, \xi_{n+m})+$$
$$+\sum_{i=n+1}^{n+m}(-1)^{\partial\xi(\partial\lambda+\partial\mu+\sum_{k=1}^{i-1}\partial\xi_k)+\partial\mu\cdot\sum_{k=1}^{n}\partial\xi_k}\lambda(\xi_1, \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, [\xi, \xi_i], \ldots, \xi_{n+m})$$
$$=(-1)^{\partial\mu\sum_{k=1}^{n}\partial\xi_k}\{\sum_{i=1}^{n}(-1)^{\partial\xi(\partial\lambda+\sum_{k=1}^{i-1}\partial\xi_k)}\lambda(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, \xi_{n+m})\}+$$
$$+(-1)^{\partial\xi\partial\lambda+(\partial\xi+\partial\mu)\sum_{k=1}^{n}\partial\xi_k}\lambda(\xi_1, \ldots, \xi_n)\times$$

$$\times\sum_{i=k+1}^{n+m}(-1)^{\partial\xi(\partial\mu+\sum_{k=n+1}^{n+m}\partial\xi_k)}\mu(\xi_{n+1}, \ldots, [\xi, \xi_i], \ldots, \xi_{n+m})$$
$$=-\{\theta_0(\xi)\lambda\otimes\mu+(-1)^{\partial\xi\partial\lambda}\lambda\otimes\theta_0(\xi)\mu\}(\xi_1, \ldots, \xi_{n+m}). \tag{4.29}$$

Now (4.17) follows from (4.19) and (4.3) since $\theta_0(\xi)$ (defined on $\mathscr{L}^n(L, V)$) commutes with A_n. To prove this, it is enough to check that $\theta_0(\xi)$ commutes with the action of the transposition $\sigma_k=\{\xi_k\leftrightarrow\xi_{k+1}\}$. Now, with $\theta^i(\xi)$ the ith term in the r.h.s. of (2.14) σ_k leaves all $\theta^i(\xi)$ with $i\neq k$, $i\neq k+1$ unaffected and exchanges $\theta^k(\xi)$ with $\theta^{k+1}(\xi)$.

We now check (4.20), from which (4.17) then follows, since $\varrho(\xi)$ evidently commutes with all σ_n, $\sigma\in\Sigma_n$. We have, for $\xi_1, \ldots, \xi_{n+m}\in L^\cdot$ using (4.13)

$$\{\varrho_3(\xi)(\lambda_1\otimes\lambda_2)\}(\xi_1, \ldots, \xi_{n+m})$$
$$=(-1)^{\partial\lambda_2\sum_{k=1}^{n}\partial\xi_k}\varrho_3(\xi)\{\lambda_1(\xi_1, \ldots, \xi_n)\cdot\lambda_2(\xi_{n+1}, \ldots, \xi_{n+m})\}$$
$$=(-1)^{\partial\lambda_2\sum_{k=1}^{n}\partial\xi_k}\{\varrho_1(\xi)\lambda_1\}(\xi_1, \ldots, \xi_n)\cdot\lambda_2(\xi_{n+1}, \ldots, \xi_{n+m})+$$
$$+(-1)^{\partial\lambda_2\sum_{k=1}^{n}\partial\xi_k+\partial\xi(\partial\lambda+\sum_{k=1}^{n}\partial\xi_k)}\lambda_1(\xi_1, \ldots, \xi_n)\cdot\varrho_2(\xi)\lambda_2(\xi_{n+1}, \ldots, \xi_{n+m})$$
$$=\{(\varrho_1(\xi)\lambda_1)\otimes\lambda_2+(-1)^{\partial\xi\partial\lambda}\lambda_1\otimes\varrho(\xi)\lambda_2\}(\xi_1, \ldots, \xi_{n+m}). \tag{4.30}$$

Now (4.14) follows from (4.17) and (4.20). We now prove (4.15) by induction w.r.t. $n+m$, using (4.12), (4.14) and the Cartan identity (2.39): we have, for $\xi\in L^\cdot$

$$i(\xi)\{\delta_{\varrho_3}(\lambda_1\wedge\lambda_2)-(\delta\varrho_1\lambda_1)\wedge\lambda_2-(-1)^n\lambda_1\wedge\delta_{\varrho_2}\lambda_2\}$$
$$=\{\theta_{\varrho_3}(\xi)-\delta_{\varrho_3}i(\xi)\}(\lambda_1\wedge\lambda_2)-\{i(\xi)\delta_{\varrho_1}\lambda_1\}\wedge\lambda_2-(-1)^{n+1+\partial\xi\partial\lambda_1}(\delta_{\varrho_2}\lambda_1)\wedge i(\xi)\lambda_2-$$
$$-(-1)^n i(\xi)\lambda_1\wedge\delta_{\varrho_2}\lambda_2-(-1)^{n+m+\partial\xi\partial\lambda_1}\lambda_1\wedge i(\xi)\delta_{\varrho_2}\lambda_2$$
$$=\{\theta_{\varrho_1}(\xi)\lambda_1\}\wedge\lambda_2+(-1)^{\partial\xi\partial\lambda_1}\lambda_1\wedge\theta_{\varrho_2}(\xi)\lambda_2-\delta_{\varrho_3}\{i(\xi)\lambda_1\wedge\lambda_2+(-1)^{n+\partial\xi\partial\lambda_1}\times$$
$$\times\lambda_1\wedge i(\xi)\lambda_2\}-\{i(\xi)\delta_{\varrho_1}\lambda_1\}\wedge\lambda_2+(-1)^{n+\partial\xi\partial\lambda_1}(\delta_{\varrho_1}\lambda_1)\wedge i(\xi)\lambda_2-$$
$$-(-1)^n i(\xi)\lambda_1\wedge\delta_{\varrho_2}\lambda_2-(-1)^{\partial\xi\partial\lambda_1}\lambda_1\wedge i(\xi)\delta_{\varrho_2}\lambda_2=0. \tag{4.31}$$

Since ξ is arbitrary, we see that (4.15) holds for $n+m=k+1$ if it holds for k. Now (4.15) holds for $n=m=0$, because for $X_1\in\Lambda^0(L, V_1)=V_1$, $X_2\in\Lambda^0(L, V_2)=V_2$

$$\{\varrho_3\wedge(X_1\wedge X_2)\}(\xi)=\{\varrho_3\wedge(X_1\cdot X_2)\}(\xi)=(-1)^{\partial\xi(\partial X_1+\partial X_2)}\varrho_3(\xi)(X_1\cdot X_2)$$
$$=(-1)^{\partial\xi(\partial X_1+\partial X_2)}\{\varrho_1(\xi)X_1\}\cdot X_2+(-1)^{\partial\xi\partial X_2}X_1\cdot\varrho_2(\xi)X_2$$
$$=(-1)^{\partial\xi\partial X}(\varrho_1\wedge X_1)(\xi)\cdot X_2+X_1\cdot(\varrho_2\wedge X_2)(\xi)$$
$$=\{(\varrho_1\wedge X_1)\wedge X_2+X_1\wedge(\varrho_2\wedge X_2)\}(\xi). \tag{4.32}$$

Setting $\varrho = 0$ in the preceding calculation, we obtain the proof of (4.21), without recourse to (4.19). Finally (4.16) follows from (4.15) via (2.34): we have

$$\Omega_3 \wedge (\lambda_1 \wedge \lambda_2) = \delta^2_{\varrho_3}(\lambda_1 \wedge \lambda_2) = \delta_{\varrho_3}((\delta_{\varrho_1}\lambda_1) \wedge \lambda_2 + (-1)^{n_1}\lambda_1 \wedge \delta_{\varrho_2}\lambda_2)$$
$$= (\delta^2_{\varrho_1}\lambda_1) \wedge \lambda_2 + \lambda_1 \wedge \delta^2_{\varrho_2}\lambda_2. \tag{4.33}$$

Returning now the case of a general left A-module for which we chose a V-connection, we shall describe the module-derivation properties of the corresponding classical operators. To this end we prefer to look at $\Lambda^*(L, V)$ (as explained in Appendix A, cf. (A.40)) as a right $\Lambda^*(L, A)$-module, and we consider $\Lambda^*(L, A)$ as a right $\Lambda^*(L, A)$-module (cf. (4.34) below). The classical operators are then derivations of this module according to the general

4.5. Definition. Let $\mathscr{A}$ be a $\mathbf{Z}/2$-graded complex algebra, with $\mathscr{E}$ a graded right $\mathscr{A}$-module; and let δ be a derivation of $\mathscr{A}$ (in the $\mathbf{Z}/2$-graded sense, cf. (A.22) in Appendix A).

A *δ-derivation of grade p* is a linear map D of grade p fulfilling

$$D(X\alpha) = (DX)\alpha + (-1)^{p\partial X} X\delta\alpha \tag{4.34}$$

for all $X \in \mathscr{E}$ of grade ∂X and $\alpha \in \mathscr{A}$.

This will apply to our situation, making $\mathscr{E} = \Lambda^*(L, V)$ and $\mathscr{A} = \Lambda^*(L, A)$ in

4.6. Corollary. *With (L, A) a graded Lie–Cartan pair, we consider the classical operators attached to a V-connection ϱ, V, a graded left A-module (cf. Definition 2.1). Moreover,*

(i) *we consider V as a right A-module (cf.* (A.40));

(ii) *we equip $\Lambda^*(L, A)$ with the graded wedge product $\wedge$ obtained by specializing* (4.3) *to the case $V = A$ (with its graded-commutative product) and with the classical operators defined in Definition* 2.1 *where we make $V = A$ and $\varrho(\xi) = \xi$, $\xi \in L$*[28];

(iii) *we consider $\Lambda^*(L, V)$ as a right $\Lambda^*(V, A)$-module by the formulae* (4.2), (4.3), *setting $V = V$, $V' = A$, $V'' = V$.*

We then have the following derivation properties for the classical operators: for $\xi \in L$, $\lambda \in \Lambda^n(E, V)$, $\alpha \in \Lambda^n(E, A)$:

$$i(\xi)\{\lambda\alpha\} = \{i(\xi)\lambda\}\alpha + (-1)^{n+\partial\xi\partial\lambda}\lambda\{i(\xi)\alpha\}, \tag{4.37}$$

$$\varrho(\xi)\{\lambda\alpha\} = \{\varrho(\xi)\lambda\}\alpha + (-1)^{\partial\xi\partial\lambda}\lambda\{\varrho(\xi)\alpha\}, \tag{4.38}$$

$$\theta_\varrho(\xi)\{\lambda\alpha\} = \{\theta_\varrho(\xi)\lambda\}\alpha + (-1)^{\partial\xi\partial\lambda}\lambda\{\theta_\varrho(\xi)\alpha\}, \tag{4.39}$$

[28] $\Lambda^*(L, A)$ is an associative algebra of (4.2) for which classical operators are derivations of the type (4.4) and we have $\delta^2 = 0$, cf. Remark 2.2.

$$\delta_\varrho\{\lambda\alpha\} = \{\delta_\varrho\lambda\}\alpha + (-1)^n\lambda\{\delta_\varrho\alpha\}, \tag{4.40}$$

$$\Omega\{\lambda\alpha\} = \{\Omega\lambda\}\alpha + \lambda\{\Omega\alpha\}. \tag{4.41}$$

In fact one has separately,

$$\theta_0(\xi)\{\lambda\alpha\} = \{\theta_0(\xi)\lambda\}\alpha + (-1)^{\partial\xi\partial\alpha}\lambda\{\theta_0(\xi)\alpha\}, \tag{4.42}$$

$$\varrho(\xi)\{\lambda\alpha\} = \{\varrho(\xi)\lambda\}\alpha + (-1)^{\partial\xi\partial\lambda}\lambda\{\varrho(\xi)\alpha\} \tag{4.43}$$

and

$$\delta_0\{\lambda\alpha\} = \{\delta_0\lambda\}\alpha + (-1)^n\lambda\{\delta_0\alpha\}, \tag{4.44}$$

$$\varrho \wedge \{\lambda\alpha\} = \{\varrho \wedge \lambda\}\alpha + (-1)^n\lambda\{\varrho \wedge \alpha\}. \tag{4.45}$$

Proof: These formulae are straightforward consequences of Proposition 4.4.

REFERENCES

[1] D. Kastler and R. Stora: *J.G.P.* **2** (3) (1985), 1–31.

[2] M. Batchelor: *Graded Manifold and Supermanifolds. Mathematical Aspects of Superspace*, Seifert, Clarke, Rosenblum (eds.), Reidel, Hamburg, July 1983.

[3] A. Rogers: *Integration on Supermanifolds*, ibid.

[4] M. Batchelor: *Transactions Amer. Math. Soc.* **258** (1) (1980), 257.

[5] Th. Schmitt: *Superdifferential Geometry*, Akad. der Wissenschaften der DDR, Institut für Mathematik.

[6] V. Molotkov: *Infinite Dimensional Z_2^k-Supermanifolds*, Trieste IC/84/183.

[7] M. Batchelor: *An Algebraic Approach to Integration*, Dept. of Pure Mathematics and Mathematical Statistics, Cambridge, England.

[8] B. De Witt: *Supermanifolds*, Cambridge University Press, Cambridge 1983.

[9] S. Rempel and T. Schmitt: *Pseudo-differential Operators and the Index Theorem on Supermanifolds*, Berlin.

APPENDIX A. GRADED VECTOR SPACES, GRADED HOMOMORPHISMS, GRADED ALGEBRAS, GRADED MODULES

Throughout this appendix all vector spaces, or algebras are either real or complex; with one of these alternatives holding throughout. Graded means throughout $\mathbf{Z}/2$-graded.

A1. Graded vector spaces

A *graded vector space* is a vector E, together with a direct sum decomposition

$$E = E^0 \oplus E^1 \tag{A.1}$$

in two vector subspaces E^0, E^1 consisting of the *even*, resp. *odd* elements of E.

Elements of $E^0 \cup E^1$ are called *homogeneous*, the *degree of* $\xi \in E^0 \cup E^1$ is by definition

$$\partial\xi = \begin{cases} 0 \mod \mathbf{Z}/2 & \text{if } \xi \in E^0, \\ 1 \mod \mathbf{Z}/2 & \text{if } \xi \in E^1 \end{cases} \tag{A.2}$$

and we set, for $i \in \mathbf{Z}$

$$E^i = \begin{cases} E^0 & \text{if } i = 0 \mod 2, \\ E^1 & \text{if } i = 1 \mod 2. \end{cases} \tag{A.3}$$

We denote furthermore by $E^{\cdot}$ the set of homogeneous elements of E

$$E^0 = E^0 \cup E^1. \tag{A.4}$$

Given graded vector spaces E, F, we define the set of *homomorphisms* $\mathrm{Hom}(E, F)$ *from* E *to* F as the vector space of linear maps: $E \to F$ equipped with the grading

$$\begin{aligned} \mathrm{Hom}(E, F)^0 &= \{a \in \mathrm{Hom}(E, F);\ aE^i \subset F^i,\ i \in \mathbf{Z}/2\}, \\ \mathrm{Hom}(E, F)^1 &= \{a \in \mathrm{Hom}(E, F);\ aE^i \subset F^{i+1},\ i \in \mathbf{Z}/2\}. \end{aligned} \tag{A.4a}$$

The usual notion of vector space appears as the specialization of graded vector spaces corresponding to *trivial grading*, i.e. $E = E^0$, $E^1 = \{0\}$. Considering $\mathbf{R}(\mathbf{C})$ as trivially graded, the *dual* of the graded vector space E is then $E^* = \mathrm{Hom}(E, \mathbf{R})$ $(= \mathrm{Hom}(E, \mathbf{C}))$, in other terms E^{*0} and E^{*1} are the respective annihilators of E^1, resp. E^0 in E^*.

Note that, since we have, with E, F, G graded vector spaces

$$\mathrm{Hom}(E, G) \subset \mathrm{Hom}(F, G) \circ \mathrm{Hom}(E, F) \tag{A.5}$$

we see that the graded vector spaces together with their homomorphisms (with composition of homomorphisms as a product) build a category.

With E, F graded vector spaces their *tensor product* $E \otimes F$ is that of E, F as vector spaces, taken as a graded vector space with the grading

$$(E \otimes F)^k = \bigoplus_{i+j=k} E^i \otimes E^j. \tag{A.6}$$

Now with E, F, E', F' graded vector spaces and $S \in \mathrm{Hom}(E, E')$, $T \in \mathrm{Hom}(F, F')$ we define *graded tensor product* $S \otimes T \in \mathrm{Hom}(E \otimes F, E' \otimes F')$ by

$$(S \otimes T)(\xi \otimes \eta) = (-1)^{\partial T \partial\xi} S\xi \otimes T\eta, \qquad \xi \in E^{\cdot},\ \eta \in F. \tag{A.7}$$

This entails, for $S' \in \mathrm{Hom}(E', E'')^{\cdot}$; $T' \in \mathrm{Hom}(F', F'')^{\cdot}$ (E'', F'' graded vector spaces), the composition rule

$$(S' \otimes T')(S \otimes T) = (-1)^{\partial T \partial S} S'S \otimes T'T \tag{A.8}$$

where we have the following formula for graded commutators:

$$\begin{gathered} [S \otimes T, S' \otimes T'] = (-1)^{\partial T \partial S'} SS' \otimes [T, T'], \\ \text{if } S'S = (-1)^{\partial S \partial S'} SS', \qquad S, S' \in \mathrm{Hom}(E, E)^{\cdot}, \qquad T, T' \in \mathrm{Hom}(F, F)^{\cdot}. \end{gathered} \tag{A.9}$$

With E_1, E_2, E_3 graded vector spaces, we shall identify $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$ (denoted $E_1 \otimes E_2 \otimes E_3$) by making the identification [1]

$$\xi_1 \otimes (\xi_2 \otimes \xi_3) = (\xi_1 \otimes \xi_2) \otimes \xi_3, \qquad \xi_i \in E_i,\ i = 1, 2, 3. \tag{A.10}$$

This matches with the definition (A.7) in as much as one has, for $S_i \in \mathrm{Hom}(E_i, F_i)$, $i = 1, 2, 3$

$$S_1 \otimes (S_2 \otimes S_3) = (S_1 \otimes S_2) \otimes S_3 \tag{A.11}$$

which acting on $\xi_1 \otimes \xi_2 \otimes \xi_3$, $\xi_i \in E_i$, $i = 1, 2, 3$, yields [2]

$$(S_1 \otimes S_2 \otimes S_3)(\xi_1 \otimes \xi_2 \otimes \xi_3) = (-1)^{\partial S_2 \partial \xi_1 + \partial S_3 (\partial \xi_1 + \partial \xi_2)} S_1 \xi_1 \otimes S_2 \xi_2 \otimes S_3 \xi_3. \tag{A.12}$$

With E, F, graded vector spaces, we may also identify $F \otimes E$ with $E \otimes F$ by deciding that

$$\eta \otimes \xi = (-1)^{\partial\xi \partial\eta} \xi \otimes \eta, \qquad \xi \in E^{\cdot},\ \eta \in F^{\cdot}. \tag{A.13}$$

With E', F' other graded vector spaces, identifying like this $F' \otimes E'$ with $E' \otimes F'$, and identifying $\mathrm{Hom}(F \otimes E, F' \otimes E')$ with $\mathrm{Hom}(E \otimes F, E' \otimes F')$, by deciding that

$$T \otimes S = (-1)^{\partial T \partial S} S \otimes T, \qquad S \in \mathrm{Hom}(E, E')^{\cdot},\ T \in \mathrm{Hom}(F, F')^{\cdot} \tag{A.14}$$

we then have that

$$(T \otimes S)(\eta \otimes \xi) = (-1)^{\partial T \partial S + \partial\eta \partial\xi} (S \otimes T)(\xi \otimes \eta). \tag{A.15}$$

These identifications allow us to alter the order of the factors in tensor products according to convenience.

We close this section by noting that, with E and F graded vector spaces, the set $\mathscr{L}^n(E, F)$, of *F-valued n-linear forms on E*, identifies with $\mathrm{Hom}(E^{\otimes n}, F)$ by requiring

$$\lambda(\xi_1, \ldots, \xi_n) = \lambda(\xi_1 \otimes \ldots \otimes \xi_n), \qquad \xi_1, \ldots, \xi_n \in E \tag{A.16}$$

with the implication that

$$\partial\{\lambda(\xi_1, \ldots, \xi_n)\} = \partial\lambda + \sum_{i=1}^{n} \partial\xi_i, \qquad \lambda \in \mathscr{L}^n(E, F)^{\cdot},\ \xi_1, \ldots, \xi_n \in E^{\cdot}. \tag{A.17}$$

[1] Identification compatible with the graded vector space structure.

[2] We then write $\xi_1 \otimes \xi_2 \otimes \xi_3$ and $S_1 \otimes S_2 \otimes S_3$ to mean the common value (A.9), resp. (A.10)

A.2 Graded algebras

A *graded algebra* is a graded vector space $A = A^0 \oplus A^1$ with a bilinear product $A \times A \to A$ such that

$$A^i A^j \subset A^{i+j}, \quad i, j \in \mathbf{Z}/2. \tag{A.18}$$

The graded algebra A is *associative* whenever

$$a(bc) = (ab)c, \quad a, b, c \in A, \tag{A.19}$$

it is *graded commutative* whenever it is associative and such that

$$ba = (-1)^{\partial a \partial b} ab, \quad a, b \in A^{\cdot}. \tag{A.20}$$

A *graded Lie algebra*[3] is a graded algebra L whose product, called the *bracket* and denoted $(\xi, \eta) \mapsto [\xi, \eta] \in L$ is such that

$$\begin{gathered}[\eta, \xi] = (-1)^{\alpha\beta+1}[\xi, \eta],\\ (-1)^{\alpha\gamma}[[\xi, \eta], \zeta]] + (-1)^{\beta\alpha}[[\eta, \zeta], \xi]] + (-1)^{\gamma\beta}[[\zeta, \xi], \eta] = 0,\\ \xi \in L^\alpha, \quad \eta \in L^\beta, \quad \zeta \in L^\gamma.\end{gathered} \tag{A.21}$$

Note that, with E a graded vector space, $\mathrm{End}(E) = \mathrm{Hom}(E, E)$ with the grading (A.4) is a graded commutative algebra under the composition of endomorphisms, and it is a graded Lie algebra under the graded commutator — bilinear extension of

$$[a, b] = ab - (-1)^{\partial a \partial b} ba, \quad a, b \in \mathrm{End}(E)^{\cdot}. \tag{A.22}$$

In fact, each associative graded algebra A becomes a graded Lie algebra (called the *commutator Lie algebra of* A) under the graded commutator defined as in (A.22) for $a, b \in A^{\cdot}$.

With A a graded algebra, the graded vector space $\mathrm{Der}\, A$ of *graded derivations of* A is defined as

$$\mathrm{Der}\, A = (\mathrm{Der}\, A)^0 \oplus (\mathrm{Der}\, A)^1 \tag{A.23}$$

with, for $i \in \mathbf{Z}/2$

$$(\mathrm{Der}\, A)^i = \{\xi \in \mathrm{End}(A)^i \colon \xi(ab) = (\xi a)b + (-1)^{ip} a(\xi b)\}, \quad a \in A^p,\ b \in A, \tag{A.24}$$

where $\mathrm{End}(A)$ denotes the set of endomorphisms of A as a vector space. With this definitions, $\mathrm{Der}\, A$ is a sub-graded Lie algebra of the commutator Lie algebra of $\mathrm{End}(A)$. If furthermore A is graded commutative, the composition $a\xi$ of $\xi \in (\mathrm{Der}\, A)^k$ and multiplication from the right by $a \in A^p$

$$(a\xi)b = a(\xi b), \quad b \in A \tag{A.25}$$

[3] Or *Lie super algebra.*

belongs to $(\mathrm{Der}\, A)^{p+k}$, and we have that

$$\begin{gathered}[\xi, a\eta] = (-1)^{pk} a[\xi, \eta] + (\xi a)\eta,\\ \xi = (\mathrm{Der}\, A)^k, \quad \eta \in \mathrm{Der}\, A, \quad a \in A^p,\end{gathered} \tag{A.26}$$

so that $(A, \mathrm{Der}\, A)$ is a graded Lie–Cartan pair, cf. Definition 1.1.

Given two graded algebras A and B, their *skew tensor product* $A \otimes B$ is, as a vector space, the usual tensor product of A and B as vector spaces, equipped with the grading

$$(A \otimes B)^n = \sum_{i+j=n} A^i \otimes B^j \tag{A.27}$$

and with the *skew product*, bilinear extension of[4]

$$\begin{gathered}(a \otimes b)(a' \otimes b') = (-1)^{\partial b \partial a'} aa' \otimes bb',\\ a \in A, \quad b \in B^{\cdot}, \quad a' \in A^{\cdot}, \quad b' \in B.\end{gathered} \tag{A.28}$$

The skew product is associative in the sense that

$$A \otimes (B \otimes C) \sim (A \otimes B) \otimes C \tag{A.29}$$

as graded algebra, for A, B, C graded algebras.

Furthermore, the skew product of two associative algebras is associative, the skew product of two graded commutative algebras is graded commutative; and the skew product of a graded commutative algebra and of a graded Lie algebra (in either order) is a graded Lie algebra.

Given a graded associative algebra A, one defines as follows the graded associative algebra $\tilde{A}$ *obtained from* A *by adjoining a unit* $\mathit{1}$. We set

$$\tilde{A} = \mathbf{R} \oplus A \quad (\mathbf{C} \oplus A) \tag{A.30}$$

(direct sum of vector spaces), with the grading

$$\begin{gathered}\tilde{A}^0 = \mathbf{R} \oplus A^0 \quad (\mathbf{C} \oplus A^0),\\ \tilde{A}^1 = \mathbf{O} \oplus A^1\end{gathered} \tag{A.31}$$

and the product given by

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab), \tag{A.32}$$

which is bilinear associative, graded in the sense (A.6), and has the unit $\mathit{1} = (1, 0)$, so that one can write

$$(\alpha, a) = \alpha \mathit{1} + a. \tag{A.33}$$

[4] Note that we would obtain (A.26) from (A.6) by letting a, a' and b, b' act by multiplication from the left on A, resp. B, and constructing $a \otimes b \in \mathrm{End}(A \otimes B)$ from $a \in \mathrm{End}(A)$, $b \in \mathrm{End}(B)$.

Note that $A \sim O \oplus A$ becomes an ideal in $\tilde{A}$, generated by the idempotent $(0, e)$ if A was unital with unit e to begin with.

A.3 Linear modules over graded commutative algebras

Throughout this section A denotes a graded commutative algebra (as a special case, with trivial grading, we have $A = \boldsymbol{R}$ ($A = \boldsymbol{C}$)).

A *graded linear left A-module* (resp. *right A-module*) E is a graded vector space $E = E^0 + E^1$ endowed with a bilinear product

$$A \times E \ni (a, \xi) \mapsto a\xi \in E, \qquad \text{(resp. } E \times A \ni (\xi a) \mapsto \xi a \in E) \tag{A.34}$$

fulfilling

$$A^p E^k \subset E^{p+k}, \quad \text{(resp. } E^k A^p \subset E^{p+k}), \qquad p, k \in \boldsymbol{Z}/2 \tag{A.35}$$

and

$$a(b\xi) = (ab)\xi, \quad a, b \in A, \ \xi \in E \qquad \text{(resp. } (\xi a) b = \xi(ab)). \tag{A.36}$$

If A is unital with unit $\mathit{1}$, the module E is called *unital* whenever

$$\mathit{1}\xi = \xi, \quad \xi \in E \qquad \text{(resp. } \xi \mathit{1} = \xi). \tag{A.37}$$

Note that a graded linear left (right) A-module over a non-unital algebra A can be made into a unital module over $\tilde{A}$ by defining[5]

$$(\alpha \mathit{1} + a)\xi = \alpha\xi + a\xi, \quad \text{(resp. } \xi(\alpha \mathit{1} + a) = \alpha\xi + \xi a), \quad \alpha \mathit{1} + a \in \tilde{A}, \quad \xi \in E. \tag{A.38}$$

Recall that, with A and B (graded commutative) algebras a *linear left B-, right A-, bimodule*[6] E is a graded vector space which is both a left B-module and a right A-module in such a way that

$$(b\xi)a = b(\xi a), \quad \xi \in E, \ a \in A, \ b \in B. \tag{A.39}$$

[5] This is the reason why we consider linear left (right) modules (aksed to be vector spaces) rather than just left (right) modules defined in terms of Abelian groups.

[6] A linear left A-right A-bimodule is called a linear A-bimodule.

With such an E and a left A-module F the *tensor product* $E \otimes_A F$ *of* E *and* F *over* A is then defined as the quotient of $E \otimes F$ through the vector space generated by the elements

$$\xi a \otimes \eta - \xi \otimes a\eta, \quad \xi \in F, \ \eta \in E, \ a \in A. \tag{A.40}$$

$E \otimes_A F$ then becomes a left B-module, left multiplication by b arising by linear extension from[7]

$$b(\xi \otimes \eta) = b\xi \otimes \eta. \tag{A.41}$$

Suppose now A is graded commutative, each linear left A-module (or linear right A-module) E is automatically a linear A-bimodule, by defining

$$\xi a \doteq (-1)^{\partial a \partial \xi} a\xi, \quad a = A^{\cdot}, \ \xi \in E^{\cdot}; \tag{A.42}$$

indeed, for $b \in A^{\cdot}$

$$(b\xi)a = (-1)^{\partial a(\partial b + \partial \xi)} ab\xi = (-1)^{\partial a \partial \xi} ba\xi = b(\xi a) \tag{A.43}$$

and

$$(\xi a)b = (-1)^{\partial b(\partial \xi + \partial a)} b(\xi a) = (-1)^{\partial b \partial a + \partial \xi(\partial a + \partial b)} ba\xi = (-1)^{\partial \xi \partial(ab)} ab\xi = \xi(ab). \tag{A.44}$$

This allows us, given linear left A-modules E and F, to build the tensor product $E \otimes_A F$ as before, or to consider the tensorial powers $E \otimes_A E \otimes_A \ldots \otimes_A E$, with the property

$$\xi_1 \otimes \ldots \otimes \xi_{i-1} \otimes a\xi_i \otimes \xi_{i+1} \otimes \ldots \otimes \xi_n = (-1)^{\partial a(\partial \xi_1 + \ldots + \partial \xi_{i-1})} a\xi_1 \otimes \ldots \otimes \xi_n, \quad a \in A^{\cdot}, \quad \xi_1, \ldots, \xi_n \in E^{\cdot}. \tag{A.45}$$

If, for arbitrary linear left A-modules, E, F we define $\mathrm{Hom}_A(E, F)$ as the graded subspace of $\mathrm{Hom}(E, F)$ (as vector spaces), given by

$$\mathrm{Hom}_A(E, F)^k = \{\lambda \in \mathrm{Hom}(E, F)^k \colon \lambda(a\xi) = (-1)^{\partial \lambda \partial a} a\lambda\xi, \ a = A^{\cdot}, \ \xi \in E\} \tag{A.46}$$

we then have that $\mathrm{Hom}_A(E^{\otimes n}, F)$ identifies, as a graded vector space, with the set $\mathscr{L}^n_A(E, F)$ of F-valued graded *n-A-linear forms on* E, setting

$$\lambda(\xi_1, \ldots, \xi_n) = \lambda(\xi_1 \otimes \ldots \otimes \xi_n), \quad \xi_1, \ldots, \xi_n \in E. \tag{A.47}$$

The elements of $\mathscr{L}^n_A(E, F)^k$ are thereby characterized as the $\lambda \in \mathscr{L}^n(E, F)$ fulfilling, for $a \in A^{\cdot}$, $\xi_1, \ldots, \xi_n \in E^{\cdot}$

$$\lambda(\xi_1, \ldots, \xi_{i-1}, a\xi_i, \xi_{i+1}, \ldots, \xi_n) = (-1)^{\partial a(\partial \lambda + \partial \xi_1 + \ldots + \partial \xi_{i-1})} a\lambda(\xi_1, \ldots, \xi_n). \tag{A.48}$$

We shall write $\mathscr{L}^n(E, F)$ to mean $\mathscr{L}^n_{\boldsymbol{R}}(E, F)$ ($\mathscr{L}^n_{\boldsymbol{C}}(E, F)$).

[7] In $E \otimes_A F$ we write $\xi \otimes \eta$ for $\xi \otimes_A \eta$.

APPENDIX B. TWISTED AND GRADED ALTERNATION THE SPACE, THE ALGEBRAS

B.1 DEFINITIONS AND NOTATION. I_n denotes the set of the first n integers and Σ_n the group of permutations of I_n.

(i) Given a set E, we denote by E^n the set of maps $I_n \to E$

$$i \in I_n \to \xi_i \in E \tag{B.1}$$

alternatively denoted by listing the ξ_i with increasing indexes

$$\xi = \{\xi_1, \dots, \xi_n\}. \tag{B.1a}$$

Setting $\xi\sigma = \xi \circ \sigma$, $\xi \in E^n$, $\sigma \in \Sigma_n$, we get a right action of Σ_n on E^n:

$$\begin{gathered}(\xi, \sigma) \in E^n \times \Sigma_n \to \xi\sigma \in E^n, \\ (\xi\sigma)\tau = \xi(\sigma\tau), \quad \xi \in E^n, \quad \sigma, \tau \in \Sigma_n, \\ \xi \,\mathrm{id} = \xi, \quad \xi \in E^n.\end{gathered} \tag{B.2}$$

The Cartesian product $E^n \times \Sigma_n$ thereby acquires a groupoid structure, for which (ξ, σ) and (η, τ) are multiplyable, with product $(\xi, \sigma\tau)$ whenever $\xi\sigma = \eta$. With K a commutative field, and $K^* = K - \{0\}$, we denote by $X^n(E)$ the set of K^*-valued characters of the groupoid $E^n \times \Sigma_n$, i.e. the set of maps $\chi\colon E^n \times \Sigma_n \to K^*$ fulfilling[8]

$$\begin{gathered}\chi(\xi, \sigma\tau) = \chi(\xi, \sigma)\chi(\xi\sigma, \tau), \\ \chi(\xi, \mathrm{id}) = 0, \quad \xi \in E^n, \ \sigma, \tau \in \Sigma_n.\end{gathered} \tag{B.3}$$

For $n, m \in N$, $\xi' \in E^n$, $\xi'' \in E^m$ we define $\xi' \otimes \xi'' \in E^{n+m}$ as follows:

$$\begin{gathered}\xi_i = \xi'_i, \quad i \in I_n, \\ \xi_{n+j} = \xi''_j, \quad j \in I_m.\end{gathered} \tag{B.4}$$

A hierarchy of characters $n \in N \mapsto \chi_n \in X^n(E)$ is then called *tensorial* whenever one has

$$\begin{gathered}\chi_{n+m}(\xi' \otimes \xi'', \sigma \otimes \mathrm{id}) = \chi_n(\xi', \sigma), \quad \xi' \in E^n, \ \xi'' \in E^m, \\ \chi_{n+m}(\xi' \otimes \xi'', \mathrm{id} \otimes \tau) = \chi_m(\xi'', \tau), \quad \sigma \in \Sigma_n, \ \tau \in \Sigma_m,\end{gathered} \tag{B.5}$$

with, for $n = 0$,

$$\chi_0(0, 0) = 1. \tag{B.6}$$

[8] Note that we then have $\chi(\xi, \tau^{-1}) = \chi(\xi\tau^{-1}, \tau)$ and that $X^n(E)$ is a group for the pointwise product.

We then define

$$(\sigma_n f)(\xi) = \chi_n(\xi, \sigma) f(\xi\sigma), \quad \xi \in E^n, \tag{B.7}$$

i.e.

$$(\sigma_n f)(\xi_1, \dots, \xi_n) = \chi_n(\xi, \sigma) f(\xi_{\sigma_1}, \dots, \xi_{\sigma_n}), \quad \xi_1, \dots, \xi_n \in E \tag{B.7a}$$

with average over Σ_n

$$A_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_n \tag{B.8}$$

(called the nth *antisymmetrizer*).

B.2 THEOREM. *With the definitions and notation of* B.1 *we have that*

(i) $\sigma \mapsto \sigma_n$ *is a linear representation of* Σ_n *on* $\mathscr{F}^n$: *One has*

$$\sigma_n \circ \tau_n = (\sigma\tau)_n, \quad (\sigma_n)^{-1} = (\sigma^{-1})_n, \quad \sigma, \tau \in \Sigma_n. \tag{B.9}$$

(ii) *One has*

$$\tau_n A_n = A_n \tau_n = A_n, \quad \tau \in \Sigma_n, \tag{B.10}$$

i.e. for $f \in \mathscr{F}^n$, $\xi_1, \dots, \xi_n \in E$

$$(A_n f)(\xi_{\tau_1}, \dots, \xi_{\tau_n}) = \chi_n(\xi, \tau)(A_n f)(\xi_1, \dots, \xi_n). \tag{B.11}$$

(iii) A_n *is an idempotent of* $\mathscr{F}^n$

$$A_n^2 = A_n \tag{B.12}$$

with range of the fixpoints of $\mathscr{F}^n$ *under all* σ_n, $\sigma \in \Sigma_n$:

$$A_n \mathscr{F}^n = \{f \in \mathscr{F}\colon f(\xi_{\tau_1}, \dots, \xi_{\tau_n}) = \chi_n(\xi, \tau) f(\xi_1, \dots, \xi_n)\}. \tag{B.13}$$

Proof: (i) One has, for $f \in \mathscr{F}^n$, $\xi_1, \dots, \xi_n \in E$, using (B.3)

$$\begin{aligned}\sigma_n(\tau_n f)(\xi_1, \dots, \xi_n) &= \chi(\xi, \sigma)(\tau_n f)(\xi_{\sigma_1}, \dots, \xi_{\sigma_n}) \\ &= \chi(\xi, \sigma)\chi(\xi\sigma, \tau) f(\xi_{\sigma\tau_1}, \dots, \xi_{\sigma\tau_n}) \\ &= \chi(\xi, \sigma\tau) f(\xi_{\sigma\tau_1}, \dots, \xi_{\sigma\tau_n}) \\ &= \{(\sigma\tau)_n f\}(\xi_1, \dots, \xi_n)\end{aligned} \tag{B.14}$$

and

$$\sigma_n(\sigma^{-1})_n = (\sigma^{-1})_n \sigma_n = (\mathrm{id})_n = \mathrm{id}. \tag{B.15}$$

(ii) One has then

$$\tau_n A_n = \frac{1}{n!} \tau_n \sum_{\sigma \in \Sigma_n} \sigma_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\tau\sigma)_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_n = A_n \tag{B.16}$$

and

$$A_n \tau_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_n \cdot \tau_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\sigma\tau)_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_n = A_n. \tag{B.17}$$

(iii) One has then

$$A_n^2 = \frac{1}{n!} \sum_{\tau \in \Sigma_n} \tau_n A_n = \frac{1}{n!} \sum_{\tau \in \Sigma_n} A_n = A_n. \tag{B.18}$$

On the other hand, for $f = A_n g$, $g \in \mathscr{F}^n$, we have, by (B.10)

$$\tau_n f = \tau_n A_n g = A_n g = f \tag{B.19}$$

and conversely, if $f \in \mathscr{F}^n$ fulfils $\sigma_n f = f$ for all $\sigma \in \Sigma_n$, we have

$$A_n f = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_n f = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f = f. \tag{B.20}$$

We now describe the linear variety of twisted antisymmetrization relevant to our graded Lie–Cartan pairs: the latter is obtained by taking for E and F graded A-modules, A graded-commutative algebra.

B.3 Definition. Let $A = A^0 \oplus A^1$ be a graded-commutative algebra, with $E = E^0 \oplus E^1$ and $F = F^0 \oplus F^1$ graded left A-modules. We denote by $\mathscr{L}_A^n(E, F)$ the set of graded F-valued n-A-linear forms, i.e. n-linear[9] maps λ: $E^n \to F$ fulfilling, for each $i \in I_n$, $a \in A^{\cdot}$, $\xi_k \in E^{\cdot}$, $k = 1, \ldots, n$

$$\lambda^{(s)}(\xi_1, \ldots, \xi_{i-1}, a\xi_i, \xi_{i+1}, \ldots, \xi_n) = (-1)^{\partial a(s + \partial\xi_1 + \ldots + \partial\xi_{i-1})} a\lambda^{(s)}(\xi_1, \ldots, \xi_n), \tag{B.21}$$

where $\lambda^{(s)}$ is the grade-s-component of λ.

On the other hand, de define as follows the $\mathbf{Z}/2$-valued *graded alternate character*[10] χ_n, $n \in N$. For $\sigma \in \Sigma_n$ with signature $\chi(\sigma)$ and $\xi \in (E^{\cdot})^n$ such that

$$\begin{aligned} \{\xi_1, \ldots, \xi_n\} &= \{\xi_{i_1}, \ldots, \xi_{i_p}\} \cup \{\xi_{j_1}, \ldots, \xi_{j_q}\}, \\ \xi_{i_k} &\in E^0, \quad 1 \leqslant i_1 < \ldots < i_p \leqslant n, \\ \xi_{j_l} &\in E^1, \quad 1 \leqslant j_1 < \ldots < j_q \leqslant n \end{aligned} \tag{B.22}$$

we set $\chi_n(\xi, \sigma) = \chi(\sigma)\chi_n^+(\xi, \sigma)$ with $\chi_n^+(\xi, \sigma) = (-1)^{\varrho(\xi, \sigma)}$, $\varrho(\xi, \sigma)$ the number of pairs l, $m \in I_q$ with $l < m$ and $\sigma_l > \sigma_m$.

B.4 Proposition. *With the definitions and notation in* B.1 *through* B.3, *we have that*

[9] Meaning $\mathbf{R}$-linear ($\mathbf{C}$-linear).

[10] The associated graded symmetrizer A will be referred to as the *graded antisymmetrizer*.

(i) *If $a_i \in A^{\cdot}$, where A is a graded-commutative algebra, then*

$$a_{\sigma_1} a_{\sigma_2} \ldots a_{\sigma_n} = \chi_n^+(a, \sigma) a_1 \ldots a_n. \tag{B.23}$$

(i) *χ_n is a character of the groupoid $E^n \times \Sigma_n$: $\chi_n \in X^n(E)$. Moreover the hierarchy χ_n is tensorial.*[11]

(iii) *σ_n as defined in* (B.7a) *with χ_n the graded alternate character in Definition* B.3 *leaves stable the subset $\mathscr{L}_A^n(E, \mathscr{F})$ of $\mathscr{L}(E^n, \mathscr{F})$. By restricting the σ_n to $\mathscr{L}_A^n(E, F)$, we get a representation of Σ_n by zero grade endomorphisms (still denoted σ_n) of $\mathscr{L}_A^n(E, F)$ as a graded vector space. We denote by $\Lambda_A^n(E, F)$ the corresponding set of fixpoints:*

$$\Lambda_A^n(E, F) = A_n \mathscr{L}_A^n(E, F) \tag{B.24}$$

and call the elements of $\Lambda_A^n(E, F)$ the F-valued graded alternate n-A-linear forms on E.

(iv) *With $E^{\otimes n}$ the graded nth tensor power of E,*[12] *and identifying $\mathscr{L}_A^n(E, F)$ with* $\mathrm{Hom}_A(E^{\otimes n}, F)$ *as follows:*

$$\lambda(\xi_1, \ldots, \xi_n) = \lambda(\xi_1 \otimes \ldots \otimes \xi_n), \quad \xi_1, \ldots, \xi_n \in E. \tag{B.25}$$

σ_n acting on $\mathscr{L}_A^n(E, F)$, $\sigma \in \Sigma_n$ is the transpose of σ acting as follows on $E^{\otimes n}$:

$$\sigma^n(\xi_1 \otimes \ldots \otimes \xi_n) = \chi(\xi, \sigma^{-1}) \xi_{\sigma^{-1}1} \otimes \ldots \otimes \xi_{\sigma^{-1}n}. \tag{B.26}$$

Proof: (i) Since A is graded-commutative, $\chi^+(a, \sigma)$ in (B.23) is the sign obtained as follows: decompose $a \in E^n$ as in (B.22), write σ as a product of transpositions and set $\chi^+(a, \sigma) = (-1)^s$, s the number of transpositions affecting elements a_{j_k} of odd degree: we then have $s = \varrho(a, \sigma)$.

(ii) Since χ is a character of Σ_n, $\chi_n \in X^n(E)$ follows from $\chi_n^+ \in X(E)$. Now with $a \in A^n$ and σ, $\tau \in \Sigma_n$ we have, by (B.23)

$$\begin{aligned} a_{\sigma\tau_1} \ldots a_{\sigma\tau_n} &= \chi_n^+(a, \sigma\tau) a_1 \ldots a_n \\ &= \chi_n^+(a\sigma, \tau) a_{\sigma_1} \ldots a_{\sigma_n} \\ &= \chi_n^+(a\sigma, \tau)\chi^+(a, \sigma) a_1 \ldots a_n. \end{aligned} \tag{B.27}$$

(iii) Let $\lambda \in \mathscr{L}_A^n(E, F)$. Since Σ_n is generated by transpositions of neighbouring elements of I_n, it is enough to show that, for each $k \in I_{n-1}$, one has $\lambda' \in \mathscr{L}_A^n(E, F)$, λ' the transform of λ by the transposition $(k \leftrightarrow k+1)$. Now if $\xi_i \in E^{\alpha_i}$, $i = 1, \ldots, n$ we have $\chi(\xi, k \leftrightarrow k+1) = (-1)^{1 + \alpha_k \alpha_{k+1}}$, hence, assuming λ of grade s

$$\lambda'(\xi_1, \ldots, \xi_n) = (-1)^{1 + \alpha_k \alpha_{k+1}} \lambda(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \xi_k, \xi_{k+2}, \ldots, \xi_n). \tag{B.28}$$

Now property (B.21) obviously holds for $i \leqslant k+1$ and $i \geqslant k+2$. Let $a \in A^p$. For

[11] $\chi_n^+(\xi, \sigma) = (-1)^{\varrho(\xi, \sigma)} = \chi(\sigma)\chi_n(\xi, \sigma)$ is also a character of the groupoid $E_n \times \Sigma_n$ (called the *graded symmetric character*) and the hierarchy χ_n^+ is also tensorial.

[12] Cf. Appendix A.

$i = k$, we have

$$\begin{aligned}\lambda'(\xi_1, \ldots, \xi_{k-1}, a\xi_k, \xi_{k+1}, \ldots, \xi_n) &= (-1)^{1+(\varrho+\alpha_k)\alpha_{k+1}} \lambda(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, a\xi_k, \ldots, \xi_n) \\ &= (-1)^{1+\alpha_k\alpha_{k+1}+p(s+\partial\xi_1+\ldots+\partial\xi_{k-1})} a\lambda(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \xi_k, \xi_{k+2}, \ldots, \xi_n) \\ &= (-1)^{p(s+\partial\xi_1+\ldots+\partial\xi_{k-1})} a\lambda'(\xi_1, \ldots, \xi_n) \end{aligned} \tag{B.29}$$

with $\partial\lambda' = \partial\lambda = s$; and for $i = k+1$

$$\begin{aligned}\lambda'(\xi_1, \ldots, \xi_k, a\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n) &= (-1)^{1+\alpha_k(p+\alpha_{k+1})} \lambda(\xi_1, \ldots, \xi_{k-1}, a\xi_{k+1}, \xi_k, \xi_{k+2}, \ldots, \xi_n) \\ &= (-1)^{1+\alpha_k\alpha_{k+1}+p(s+\alpha_1+\ldots+\alpha_{k-1})} a\lambda(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \xi_k, \xi_{k+2}, \ldots, \xi_n) \\ &= (-1)^{p(s+\alpha_1+\ldots+\alpha_{n-1})} a\lambda'(\xi_1, \ldots, \xi_{k-1}, \xi_k, \xi_{k+1}, \ldots, \xi_n). \end{aligned} \tag{B.30}$$

(iv) For $\lambda \in \mathscr{L}^n_A(E, F)$, $\sigma \in \Sigma_n$, $\xi \in E^n$, we have

$$\begin{aligned}(\sigma_n\lambda)(\sigma^n(\xi_1 \otimes \ldots \otimes \xi_n)) &= \chi(\xi, \sigma^{-1})(\sigma_n\lambda)(\xi_{\sigma^{-1}1}, \ldots, \xi_{\sigma^{-1}n}) \\ &= \chi(\xi\sigma^{-1}, \sigma)\chi(\xi\sigma^{-1}, \sigma)\lambda(\xi_1, \ldots, \xi_n) \\ &= \lambda(\xi_1, \ldots, \xi_n). \end{aligned} \tag{B.31}$$

B.5 *Remark.* As suggested by (B.3) we could adopt the notation

$$\chi(\xi, \sigma) = \left(\frac{\xi\sigma}{\xi}\right) \tag{B.32}$$

transforming (B.3) into the "Leibniz rule"

$$\left(\frac{\xi\sigma\tau}{\xi}\right) = \left(\frac{\xi\sigma\tau}{\xi\sigma}\right)\left(\frac{\xi\sigma}{\xi}\right). \tag{B.33}$$

B.6 PROPOSITION. *Let A be a graded commutative complex algebra, and E a graded left A-module. If, given $\lambda \in \mathscr{L}^n(E, A)$, $\mu \in \mathscr{L}^m(E, A)$, we define $\lambda \otimes \mu$ as*

$$\begin{gathered}(\lambda\otimes\mu)(\xi_1, \ldots, \xi_{n+m}) = (-1)^{\partial\mu(\partial\xi_1+\ldots+\partial\xi_n)} \lambda(\xi_1, \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, \xi_{m+n}), \\ \xi_i \in E^{\cdot}, \quad i = 1, 2, \ldots, n+m, \end{gathered} \tag{B.34}$$

then we get an associative-graded tensor product $\otimes$: one has

$$\lambda\otimes(\mu\otimes\nu) = (\lambda\otimes\mu)\otimes\nu, \quad \lambda \in \mathscr{L}^n(E, A),\ \mu \in \mathscr{L}^m(E, A),\ \nu \in \mathscr{L}^r(E, A). \tag{B.35}$$

Moreover, $\lambda\otimes\mu \in \mathscr{L}^{m+n}_A(E, A)$ if $\lambda \in \mathscr{L}^n_A(E, A)$ and $\mu \in \mathscr{L}^m_A(E, A)$.[13] *Defining further,*

[13] $\mathscr{L}^n_A(E, F)$ is defined in Appendix A (cf. (A.47), (A.48)).

for $\lambda \in \mathscr{L}^n(E, A)$, $\mu \in \mathscr{L}^m(E, A)$

$$\lambda \wedge \mu = \frac{(n+m)!}{n!\,m!} A_{n+m}(\lambda\otimes\mu) \tag{B.36}$$

with A_{n+m} the graded antisymmetrizer defined in term of the graded alternate character (*cf.* Definition B.3), *we get a graded wedge product $\wedge$ which is asociative*

$$\begin{gathered}\lambda \wedge (\mu \wedge \nu) = (\lambda \wedge \mu) \wedge \nu = \frac{(n+m+r)!}{n!\,m!\,r!} A_{n+m+r}(\lambda\otimes\mu\otimes\nu), \\ \lambda \in \mathscr{L}^n(E, A), \quad \mu \in \mathscr{L}^m(E, A), \quad \nu \in \mathscr{L}^r(E, A). \end{gathered} \tag{B.37}$$

Hence, under the bilinear extension of the product, (B.36), *$\Lambda^*(E, A)$ is an associative algebra with subalgebra $\Lambda^*_A(E, A)$.*

Proof: For $a \in A^p$, writing $\partial\lambda = s$, $\partial\mu = t$, $\partial\xi_i = \alpha_i$, $i \leqslant n$

$$\begin{aligned}(\lambda\otimes\mu)(\xi_1, \ldots, \xi_{i-1}, a\xi_i, \xi_{i+1}, \ldots, \xi_{n+m}) &= (-1)^{t(\alpha_1+\ldots+\alpha_n+p)+p(s+\alpha_1+\ldots+\alpha_{i-1})} a\lambda(\xi_1, \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, \xi_{n+m}) \\ &= (-1)^{p(s+t+\alpha_1+\ldots+\alpha_{i-1})} a(\lambda\otimes\mu)(\xi_1, \ldots, \xi_{n+m}) \end{aligned} \tag{B.38}$$

and for $i > n$

$$\begin{aligned}(\lambda\otimes\mu)(\xi_1, \ldots, \xi_{i-1}, a\xi_i, \xi_{i+1}, \ldots, \xi_{n+m}) &= (-1)^{t(\alpha_1+\ldots+\alpha_n)+p(t+\alpha_{n+1}+\ldots+\alpha_{i-1})} \lambda(\xi_1, \ldots, \xi_n)a\mu(\xi_{n+1}, \ldots, \xi_{n+m}) \\ &= (-1)^{t(\alpha_1+\ldots+\alpha_n)+p(t+s+\alpha_1+\ldots+\alpha_{i-1})} a\lambda(\xi_1, \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, \xi_{n+m}) \\ &= (-1)^{p(s+t+\alpha_1+\ldots+\alpha_{i-1})} a(\lambda\otimes\mu)(\xi_1, \ldots, \xi_{n+m}). \end{aligned} \tag{B.39}$$

We proved the A-linearity of $\lambda\otimes\mu$. We now check (B.35): we have, for $\lambda \in \mathscr{L}^n(E, A)$, $\mu \in \mathscr{L}^m(E, A)$

$$\begin{aligned}\{(\lambda\otimes\mu)\otimes\nu\}(\xi_1, \ldots, \xi_{n+m+r}) &= (-1)^{\partial\nu\sum_{i=1}^{n+m}\partial\xi_i} (\lambda\otimes\mu)(\xi_1, \ldots, \xi_{n+m})\nu(\xi_1, \ldots, \xi_{n+m+r}) \\ &= (-1)^{\partial\mu\sum_{i=1}^{n}\partial\xi_i+\partial\nu\sum_{i=1}^{n+m}\partial\xi_i} \lambda(\xi_1, \ldots, \xi_n)\mu(\xi_{n+1}, \ldots, \xi_{n+m})\nu(\xi_{n+m+1}, \ldots, \xi_{n+m+r}) \\ &= (-1)^{(\partial\mu+\partial\nu)\sum_{i=1}^{n}\partial\xi_i} \lambda(\xi_1, \ldots, \xi_n)(\mu\otimes\nu)(\xi_{n_1}, \ldots, \xi_{n+m+r}) \\ &= \{\lambda\otimes(\mu\otimes\nu)\}(\xi_1, \ldots, \xi_{n+m+r}). \end{aligned} \tag{B.40}$$

We now check (B.27) in this context, from which (B.18) where $f \to \lambda$, $g \to \mu$ follows as in the proof of Theorem B.2 (iv), entailing (B.37). We have, indeed, from (B.34), for λ, μ and the ξ_i as above

$$(\sigma_n \lambda \otimes \mu)(\xi_1, \ldots, \xi_{n+m})$$
$$= (-1)^{t(\alpha_1+\ldots+\alpha_n)} \chi(\xi', \sigma)\, \lambda(\xi_{\sigma_1}, \ldots, \xi_{\sigma_n})\, \mu(\xi_{n+1}, \ldots, \xi_{n+m})$$
$$= \chi(\xi', \sigma)(\lambda \otimes \mu)(\xi_1, \ldots, \xi_{n+m})$$

and

$$(\lambda \otimes \tau_m \mu)(\xi_1, \ldots, \xi_{n+m})$$
$$= (-1)^{t(\alpha_1+\ldots+\alpha_n)} \chi(\xi'', \tau)\, \lambda(\xi_1, \ldots, \xi_n)\, \mu(\xi_{n+1}, \ldots, \xi_{n+m})$$
$$= \chi(\xi'', \tau)(\lambda \otimes \mu)(\xi_1, \ldots, \xi_{n+m}), \tag{B.42}$$

whence our conclusion (using (B.5) we denoted $\{\xi_1, \ldots, \xi_n\} = \xi'$, $\{\xi_{n+1}, \ldots, \xi_{n+m}\} = \xi''$).

B.7 *Remark*. We note the expression of the graded wedge product of a one-form and an n-form: for $\varphi \in \Lambda^1_A(E)$, $\lambda \in \Lambda^n_A(E)^{\cdot}$, resp. $\varphi \in \Lambda^1(E, F)$, $\lambda \in \Lambda^n(E, F)^{\cdot}$, formula (B.36) reads

$$(\varphi \wedge \lambda)(\xi_1, \ldots, \xi_{n+1}) = -\sum_{i=1}^{n+1} (-1)^{i+\partial\xi_i\left(\partial\lambda+\sum_{k=1}^{i-1}\partial\xi_k\right)} \varphi(\xi_i)\, \lambda(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}), \tag{B.43}$$

where $\hat{}$ means omission.

Proof: Use the fact that $\Sigma_{n+1} = \bigcup_{i \in I_{n+1}} \Sigma_i \circ \sigma_i$ where

$$\sigma_i = \begin{pmatrix} 1 & 2 \ldots & & n+1 \\ i & 1 \ldots \hat{i} \ldots & & n+1 \end{pmatrix}, \quad 1 \leqslant i < n+1 \tag{B.44}$$

and Σ_i is the subgroup of Σ_{n+1} leaving i invariant. (B.43) then follows from $\mathrm{Card}\, \Sigma_i = n!$ and

$$\chi(\xi, \sigma_i) = (-1)^{i+1+\partial\xi_i \sum_{k=1}^{i-1} \partial\xi_k}. \tag{B.45}$$

B.8 *Remark*. Applied to the trivially graded commutative algebra $A = \boldsymbol{R}$ $(= \boldsymbol{C})$ with E a real (complex) vector space, Proposition B.6 describes the algebra $\Lambda^*(E) = \bigoplus_{n \in \boldsymbol{N}} \Lambda^n(E)$ of real (complex) graded alternate multilinear forms on E. $\Lambda^*(E)$ is a subalgebra of the algebra $A\mathscr{F}^*$ in Theorem B.2 (iv). $\Lambda^*(E)$ gives rise to the

commutation property

$$\mu \wedge \lambda = (-1)^{nm+\partial\lambda\partial\mu} \lambda \wedge \mu, \quad \lambda \in \Lambda^n(E),\ \mu \in \Lambda^m(E), \tag{B.46}$$

hence it is graded commutative for a trivially graded E.

Proof of (B.46): We first look at elements λ, $\mu \in \Lambda'(E)$.
Now we have, for $\xi_1 \in E^{\alpha_1}$, $\xi_2 \in E^{\alpha_2}$, $\partial\lambda = s$, $\partial\mu = t$

$$(\mu \wedge \lambda)(\xi_1, \xi_2) = \{A_2(\mu \otimes \lambda)\}(\xi_1, \xi_2)$$
$$= \tfrac{1}{2}\{(\mu \otimes \lambda)(\xi_1, \xi_2) - (-1)^{\alpha_1\alpha_2}(\mu \otimes \lambda)(\xi_2, \xi_1)\}$$
$$= \tfrac{1}{2}\{(-1)^{s\alpha_1} \mu(\xi_1)\lambda(\xi_2) - (-1)^{\alpha_1\alpha_2+s\alpha_2} \mu(\xi_2)(\xi_1)$$
$$= \tfrac{1}{2}\{(-1)^{st} \mu(\xi_1)\lambda(\xi_2) - \mu(\xi_2)\lambda(\xi_1)\}, \tag{B.47}$$

where we used the facts that, by definition, with δ_{ik} the Kronecker symbol

$$\mu(\xi_1) = \delta_{t\alpha_1} \mu(\xi_1), \quad \mu(\xi_2) = \delta_{t\alpha_2} \mu(\xi_2), \quad \lambda(\xi_1) = \delta_{s\alpha_1} \lambda(\xi_1)$$
$$\lambda(\xi_2) = \delta_{s\alpha_2} \lambda(\xi_2).$$

Exchange of λ and μ then shows that we have

$$\mu \wedge \lambda = (-1)^{st+1} \lambda \wedge \mu, \quad \partial\lambda = s,\ \partial\mu = t. \tag{B.48}$$

Now, by linearity, it is enough to check (B.46) for $\lambda = \varphi_1 \wedge \ldots \wedge \varphi_n$, $\mu = \psi_1 \wedge \ldots \wedge \psi_m$, φ_i, $\psi_j \in \Lambda^1(E)^{\cdot}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Applying (B.44) nm times then brings about a factor $(-1)^{nm+\left(\sum_{i=1}^{n}\partial\varphi_i\right)\left(\sum_{j=1}^{m}\partial\psi_j\right)}$.