Graded Lie–Cartan Pairs. II. The Fermionic Differential Calculus

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Received February 18, 1987

We describe ab initio the algebraic features of "fermionic differential calculus"—the graded commutative generalization of the algebraic theory of exterior (covariant) derivatives, Lie-derivatives, and interior products. Our study is centered around the (module) derivation properties of those operators.

1. INTRODUCTION

This paper is a (self-contained) sequel to our paper [2] (quoted below as [I]), in which the *Lie–Cartan pairs* of [1], describing the algebraic features of the classical operators of differential geometry, are generalized to the graded commutative frame (case of "anticommuting variables").

We here present a variant of the formalism in [I] which is both mathematically more systematic (with the algebra of differential forms graded commutative and all classical operators graded derivations for the total grading) and physically more relevant (as the algebraic extraction of the "fermionic differential calculus" having Berezin's integration as its integral calculus counterpart).

Apart from a shift in content, the present paper offers a technique of proof independent of [I]—hence yielding a self-contained exposition. We mention in addition the proofs obtained by adapting the results of [I], thus providing a welcome double check of the involved sign factors appearing in the formulae of both papers.

As will become apparent, both the classical identities relating exterior (covariant) derivatives, Lie derivatives and interior products (cf. Theorem [3.2](iii)), and the (module) derivation properties (cf. Theorems [4.1] and [5.3]), are identical with the corresponding features of usual differential geometry if written in terms of graded commutators, and if use is made of the total grading (sum of the order of the form and the intrinsic grading, the latter trivial in usual differential geometry).

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0003-4916/87 $7.50

Thus graded commutative algebras appear as the natural setting for phrasing the algorithmic aspects of differential geometry. From a spatial point of view, the graded-commutative generalization corresponds to the passage from usual differential manifolds to supermanifolds (cf. [3]). But the point of our study is to isolate algebraic aspects, dispensing with spatial constructions. In this respect we note that, while "spaces" and "commutative algebras" are categorically equivalent (specifically: locally compact spaces and abelian C^*-algebras, via the Gelfand structure theory of the latter), we do not have an analogous general "spatial implementation" of graded commutative algebras by means of super-manifolds (the reader may consult [4] for a discussion of related problems, and [5, 6] for indications of directions for further work).

We conclude this introduction with a description of the differences between the present formalism and that of [I]. The axioms of graded Lie–Cartan pairs remain the same as in [I], as well as the definition of the E-connections relative to a graded A-module E, for a given Lie–Cartan pair (L, A). The differences appear at the level of specifying the "classical operators" δ_ρ, δ_0, $\rho\wedge$, $\theta_\rho(\xi)$, $\theta_0(\xi)$, $\rho(\xi)$, $i(\xi)$, $\xi \in L$, and the spaces on which they act. The modifications are as follows:

(i) The sets $\bigwedge^*(L, A)$ (resp. $\bigwedge^*_A(L, A)$) of graded alternate A-valued $\mathbb{C}$-multilinear (resp. A-multilinear) forms on L are unchanged, but the wedge product $\alpha \wedge \beta$ is modified by addition of a sign

$$(-1)^{\text{intrinsic grade of } \alpha \cdot \mathbb{N}\text{-grade of } \beta}. \tag{1.1}$$

This has the effect of *making $\bigwedge^*(L, A)$ and $\bigwedge^*_A(L, A)$ graded commutative algebras w.r.t. the total grading.*

The definition formulae for δ_ρ and $\rho\wedge$ are unchanged. But those of $\theta_\rho(\xi)\,\alpha$, $\theta_0(\xi)\,\alpha$, $\rho(\xi)\,\alpha$, and $i(\xi)\,\alpha$ are modified by addition of a sign

$$(-1)(-1)^{\text{grade of } \xi \cdot \mathbb{N}\text{-grade of } \alpha}. \tag{1.2}$$

This has the effect of *making the classical operators graded derivations uniformly w.r.t. the total grading* (with δ and $\rho\wedge$ of grade 1, $\theta_\rho(\xi)$, $\theta_0(\xi)$, and $\rho(\xi)$ of grade $\partial\xi$, and $i(\xi)$ of grade $\partial\xi + 1$).

(ii) For E a graded A-module, and ρ an E-connection (resp. a local E-connection), the corresponding classical operators act on the (unchanged) space

$$E \otimes_{\mathbb{C}} \bigwedge^* (L, A) \tag{1.3}$$

(resp.

$$E \otimes_A \bigwedge^*_A (L, A)), \tag{1.4}$$

now a right $\bigwedge^*(L, A)$-module (resp. $\bigwedge^*_A$-module) for the new product structure of these algebras (cf. (1.1)).

Moreover, the identification of the elements $X \otimes \alpha$ of the above tensor products (1.3) (resp. (1.4)) with elements of $\bigwedge^*(L, E)$ (resp. $\bigwedge^*_A(L, E)$) is modified by interposition of a sign

$$(-1)^{\text{grade of } X \cdot \mathbb{N}\text{-grade of } \alpha}. \tag{1.5}$$

The definition formula of $\delta_\rho \lambda$ then remains unchanged, while those of $\theta_\rho(\xi)$ and $i(\xi)$ are modified by adding a sign

$$(-1)^{\text{grade of } \xi \cdot \mathbb{N}\text{-grade of } \alpha}. \tag{1.6}$$

This has the effect of *making all classical operators graded derivations of the module* (1, 3) (*resp.* (1.4)) *w.r.t. to the total grading*, namely δ_ρ a δ-derivation of grade 1, $\theta_\rho(\xi)$ a $\theta(\xi)$-derivation of grade $\partial\xi$, and $i(\xi)$ an $i(\xi)$-derivation of grade $1 + \partial\xi$.

Our exposition proceeds as follows: after defining graded Lie–Cartan pairs and their E-connections in Section 2, the classical operators attached to a given E-connection are defined in Section 3 which states in Theorem [3.2]: (i) their property of preserving graded alternation; (ii) their "locality properties" for a local connection ρ; (iii) the classical identities which they fulfill—the latter being checked in low order (on zero- and one-forms). In Section 4 we study the particular case $E = A$, show that the sets $\bigwedge^*(L, A)$ (resp. $\bigwedge^*_A(L, A)$) of graded alternate (resp. "local" graded alternate) forms are graded commutative bigraded differential algebras; and prove that the classical operators are graded derivations of those algebras (Theorem [4.1]—we give two *ab initio* proofs of the last fact). Section 5 returns to the general E-valued case, expresses the $\bigwedge^*(L, E)$-valued classical operators in terms of the $\bigwedge^*(L, A)$-valued ones (Lemma [5.2]), reduces the module-derivation properties of the first to the derivation properties of the latter (Theorem [5.3]) and uses these for proving the identities between classical operators. All these results are independently checked by adapting the results of [I] to the present modified frame.

2. Graded Lie–Cartan Pairs: E-Connections

[2.1] Definition. A pair (L, A) of a Lie superalgebra L, and a graded commutative algebra (both complex[1], the latter with unit $\mathbb{1}$) is a **graded Lie–Cartan pair** whenever

(i) we have a (grade zero) homomorphism[2]

$$\xi \in L \to d(\xi) = \{a \in A \to \xi a \in A\} \tag{2.1}$$

[1] In what follows the word "complex" could be replaced throughout by "real."

[2] In order to alleviate notation, and so as to generalize the usual notation in differential geometry, we write ξa instead of $d(\xi)\, a$.

of the Lie superalgebra L into the Lie superalgebra Der A of derivations[3] of A;

(ii) L is a graded left A-module, unital in the sense $\mathbb{1}\xi = \xi$, $\xi \in L$;

(iii) we have the properties

$$a(\xi b) = (a\xi)\, b, \qquad a, b \in A,\ \xi \in L \tag{2.3}$$

and

$$[\xi, a\eta] = (-1)^{\partial\xi\partial a} a[\xi, \eta] + (\xi a)\, \eta, \qquad \xi \in L^{\cdot}\ \eta \in L,\ a \in A^{\cdot} \tag{2.4}$$

where [ ] denotes a graded commutator and we denote by $E^{\cdot}$ the set of homogeneous elements of the graded vector space $E = E^0 \oplus E^1 (E^{\cdot} = E^0 \cup E^1)$, with ∂x the grade of $x \in E^{\cdot}$.

Given a graded unital A-module E (which we want to consider here as a right A-module[4]) an **E-connection** ρ is a zero grade $\mathbb{C}$-linear assignment of a $\mathbb{C}$-linear operator $\rho(\xi)$ in E to each $\xi \in L$, fulfilling the property

$$\rho(\xi)(Xa) = \{\rho(\xi)\, X\}\, a + (-1)^{\partial\xi\partial X} X(\xi a), \qquad \xi \in L^{\cdot}\ X \in E^{\cdot}\ a \in A. \tag{2.5}$$

The corresponding **curvature** Ω_ρ is defined by

$$\Omega_\rho(\xi, \eta) = [\rho(\xi), \rho(\eta)] - \rho([\xi, \eta]), \qquad \xi, \eta \in L, \tag{2.6}$$

where [,] denotes graded commutators[5]. The E-connection ρ is called **local** whenever it fulfills

$$\rho(a\xi)\, X = a\rho(\xi)\, X, \qquad \xi \in L,\ a \in A, \tag{2.7}$$

and **flat** whenever $\Omega_\rho = 0$ (i.e., whenever ρ is a representation of the Lie superalgebra L).

[2.2] We recall that

(i) the assumptions, made in [2.1], of the existence of a unit $\mathbb{1}$ in A is not a restriction in generality (cf. [I, [1.8]). One can drop this assumption, considering linear representation spaces E of A instead of A-modules.

(ii) each graded Lie–Cartan pair (L, A) is accompanied by its **depletion**, the degenerate[6] Lie–Cartan pair obtained by keeping all the products $a\xi$, and replacing all the products ξa by zero, $\xi \in L$, $a \in A$ (cf. [I, [1.4], [1.5], [1.6]). While the set of E-connections for a fixed E is generally an affine space, this space becomes linear (goes through the zero) for a degenerate Lie–Cartan pair.

(iii) $\Omega_\rho(\xi, \eta)$ is a homomorphism of the A-module E (of grade $\partial\xi + \partial\eta$ for ξ, $\eta \in E^{\cdot}$ cf. [I, [1.2]]).

(iv) the map d in (2.1) is a local and flat A-connection.

[2.3] There are two distinguished A-modules, namely A and L. There exists a canonical, local, and flat A-connection $\rho(\xi) \equiv d(\xi)$ which will be discussed in Section 4.

The second of the two distinguished modules, the A-module L, also has a canonical flat connection: $\mathrm{ad}(\xi) \equiv [\xi, \cdot]$. This connection is however nonlocal (unless (L, A) is degenerate). In general there exists no canonical local L-connection. For a generic L-connection ∇ the **torsion** T_∇ **of** ∇ is defined by

$$T_\nabla(\xi, \eta) = \nabla_\xi \eta - (-1)^{\partial\xi\partial\eta}\, \nabla_\eta \xi - [\xi, \eta], \qquad \eta, \xi \in L^{\cdot}.$$

The map $L \times L \ni (\xi, \eta) \mapsto T_\nabla(\xi, \eta) \in L$ is $\mathbb{C}$-bilinear graded antisymmetric. If ∇ is local then this map is also graded A-bilinear.

3. The Classical Operators Attached to an E-Connection

[3.1] Definitions. Let (L, A) be a graded Lie–Cartan pair, with E a graded unital A-module, and ρ an E-connection. And denote[7] by $\mathscr{L}^n(L, E)$ (resp. $\mathscr{L}^n_A(L, E)$) the set of E-valued $n - \mathbb{C}$-linear (resp. $n - A$-linear) forms on A. The **classical operators** δ_ρ, $\rho\wedge$, δ_0; $\theta_\rho(\xi)$, $\theta_0(\xi)$, $\rho(\xi)$; $i(\xi)$; $\Omega_\rho(\xi, \eta)$, $\Omega_\rho \wedge$; $\xi, \eta \in L^{\cdot}$ **attached to the E-connection** ρ are defined as follows on the space $\mathscr{L}^*(L, E) = \bigoplus_{n \in \mathbb{N}} \mathscr{L}^n(L, E)$: for $\lambda \in \mathscr{L}^n(L, E)$ of intrinsic grade $\partial_0\lambda$, $n \geqslant 1$, and $\xi, \eta, \xi_1, ..., \xi_{n+2} \in L^{\cdot}$ we set

$$\delta_\rho = \delta_0 + \rho\wedge \tag{3.1}$$

with[8]

$$(\delta_0\lambda)(\xi_1, ..., \xi_{n+1}) = \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{\alpha_{ij}} \lambda([\xi_i, \xi_j], \xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_{n+1}),$$

$$\text{where } \alpha_{ij} = i + j + (\partial\xi_i + \partial\xi_j) \sum_{k=1}^{i-1} \partial\xi_k + \partial\xi_j \sum_{k=i+1}^{j-1} \partial\xi_k \tag{3.2}$$

$$\delta_0 = 0 \qquad \text{on } \mathscr{L}^0(L, E) = E$$

[3] Derivations in the graded commutative sense, i.e., sums of even derivations and odd anti-derivations, with the graded commutator as the bracket. We recall that one has $\xi\mathbb{1} = 0$ for all $\xi \in \mathrm{Der}\, A$.

[4] We recall that the notions of (graded unital) right A-module and left A-module coincide for a graded commutative algebra A (via the convention $aX = (-1)^{\partial a\partial X} Xa$, $X \in E^{\cdot}$ $a \in A^{\cdot}$ which makes E a bimodule).

[5] This will be the case throughout this paper.

[6] The Lie–Cartan pair (L, A) is *degenerate* whenever $\xi a = 0$ for all $\xi \in L$ and $a \in A$.

[7] By definition $\mathscr{L}^0(L, E) = E$. See Appendix A for definitions.

[8] The caret ^ means omission of the corresponding argument.

and

$$(\rho \wedge \lambda)(\xi_1, ..., \xi_{n+1}) = \sum_{i=1}^{n+1} (-1)^{\beta_i} \rho(\xi_i) \lambda(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1}),$$
$$\text{where } \beta_i = 1 + i + \partial\xi_i \left(\partial_0 \lambda + \sum_{k=1}^{i-1} \partial\xi_k\right), \tag{3.3}$$

further

$$\theta_\rho(\xi) = \theta_0(\xi) + \rho(\xi) \tag{3.4}$$

with

$$\{\theta_0(\xi)\lambda\}(\xi_1, ..., \xi_n) = (-1)^{n\,\partial\xi + 1} \sum_{i=1}^{n} (-1)^{\gamma_i} \lambda(\xi_1, ..., \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, ..., \xi_n),$$
$$\text{where } \gamma_i = \partial\xi \left(\partial_0\lambda + \sum_{k=1}^{i-1} \partial\xi_k\right) \tag{3.5}$$
$$\theta_0(\xi) = 0 \qquad \text{on } \mathscr{L}^0(L, E) = E$$

and[9]

$$\{\rho(\xi)\lambda\}(\xi_1, ..., \xi_n) = (-1)^{n\,\partial\xi} \rho(\xi)\{\lambda(\xi_1, ..., \xi_n)\}, \tag{3.6}$$

finally

$$\{i(\xi)\lambda\}(\xi_1, ..., \xi_{n-1}) = (-1)^{(n+\partial_0\lambda)\,\partial\xi} \lambda(\xi, \xi_1, ..., \xi_{n-1})$$
$$i(\xi) = 0 \qquad \text{on } \mathscr{L}^0(L, E) = E \tag{3.7}$$

$$\{\Omega_\rho(\xi, \eta)\lambda\}(\xi_1, ..., \xi_n) = (-1)^{n(\partial\xi + \partial\eta)} \Omega_\rho(\xi, \eta)\{\lambda(\xi_1, ..., \xi_n)\} \tag{3.8}$$

$$(\Omega_\rho \wedge \lambda)(\xi_1, ..., \xi_{n+2}) = -\sum_{1 \leqslant i < j \leqslant n+2} (-1)^{\alpha_{ij} + \partial_0\lambda(\partial\xi_i + \partial\xi_j)} \times \Omega_\rho(\xi_i, \xi_j)\lambda(\xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_{n+2}). \tag{3.9}$$

The following theorem states the main properties of these classical operators—apart from their module derivation properties discussed below in Section 5, whose description relies on the algebra $\bigwedge^* (L, A)$ described in Section 4.

[3.2] THEOREM. *With (L, A), E, ρ, δ_ρ, δ_0, $\rho \wedge$, $\theta_\rho(\xi)$, $\theta_0(\xi)$, $\rho(\xi)$; $i(\xi)$; $\Omega_\rho(\xi, \eta)$, and $\Omega_\rho \wedge$ as in [3.1] we have that*

[9] Note that $\rho(\xi)$, resp. $\Omega_\rho(\xi, \eta)$, as defined in (3.6), resp. (3.8), extend $\rho(\xi)$ and $\Omega_\rho(\xi, \eta)$ originally defined on $E = \mathscr{L}^0(L, E)$.

(i) *These operators leave invariant the set $\bigwedge^* (L, E)$ of E-valued graded alternate multilinear forms on L. Specifically, with $\bigwedge^n (L, E)_k$ the set of such n-forms of intrinsic grade[10] k, $\lambda \in \bigwedge^n (L, E)$ of intrinsic grade $\partial_0\lambda$, and $\xi, \eta \in L^\cdot$ of respective grades $\partial\xi$, $\partial\eta$, we have that*

$$\delta_0\lambda,\ \rho \wedge \lambda, \qquad \delta_\rho\lambda \in \textstyle\bigwedge^{n+1} (L, E)_{\partial_0\lambda}$$
$$\theta_0(\xi)\lambda,\ \rho(\xi)\lambda, \qquad \theta_\rho(\xi)\lambda \in \textstyle\bigwedge^{n} (L, E)_{\partial_0\lambda + \partial\xi}$$
$$i(\xi)\lambda \in \textstyle\bigwedge^{n-1} (L, E)_{\partial_0\lambda + \partial\xi} \tag{3.10}$$
$$\Omega_\rho \wedge \in \textstyle\bigwedge^{n+2} (L, E)_{\partial_0\lambda}$$
$$\Omega_\rho(\xi, \eta)\lambda \in \textstyle\bigwedge^{n} (L, E)_{\partial_0\lambda + \partial\xi + \partial\eta}.$$

(ii) *$\bigwedge_A^* (L, E) = \bigoplus_{n \in \mathbb{N}} \bigwedge_A^n (L, E)$ is stable under $i(\xi)$ and $\Omega_\rho(\xi, \eta)$; and, if the connection ρ is local, under δ_ρ, $\theta_\rho(\xi)$, $\xi \in L$, and $\Omega_\rho \wedge$.*

(iii) *We have the following relations, where $[\ ,\ ]$ stands for graded commutators w.r.t. the total grading $\partial = \partial_0 + n$, and $\xi, \eta \in L$:*

$$[i(\xi), i(\eta)] = 0 \tag{3.11}$$
$$[\theta_\rho(\xi), \theta_\rho(\eta)] = \theta_\rho([\xi, \eta]) + \Omega_\rho(\xi, \eta) \tag{3.12}$$
$$\delta_\rho^2 = \Omega_\rho \wedge \tag{3.13}$$
$$[\delta_\rho, i(\xi)] = \theta_\rho(\xi) \tag{3.14}$$
$$[i(\xi), \theta_\rho(\eta)] = i([\xi, \eta]) \tag{3.15}$$
$$[\delta_\rho, \theta_\rho(\xi)] + [i(\xi), \Omega_\rho \wedge] = 0 \tag{3.16}$$
$$[\theta_0(\xi), \theta_0(\eta)] = \theta_0([\xi, \eta]) \tag{3.12a}$$
$$\delta_0^2 = 0 \tag{3.13a}$$
$$[\delta_0, i(\xi)] = \theta_0(\xi) \tag{3.14a}$$
$$[i(\xi), \theta_0(\eta)] = i([\xi, \eta]) \tag{3.15a}$$
$$[\theta_0(\xi), \delta_0] = 0. \tag{3.16a}$$

[10] See Appendix A, definition (A, 17). The **total grade** of $\lambda \in \bigwedge^n (L, E)_k$ is, by definition, $\partial\lambda = \partial_0\lambda + n = k + n$. We shall denote $\bigwedge^n (L, E)^m$ the set of graded alternate n-forms of total grade m. Thus $\bigwedge^n (L, E)_k = \bigwedge^n (L, E)^{n+k}$, $\bigwedge^n (L, E)^m = \bigwedge^n (L, E)_{m+n}$, $k, n+k, m, n+m \in \mathbb{Z}/2$. Note that δ_0, $\rho \wedge$ and δ_ρ are of total grade 1; $\theta_0(\xi)$, $\rho(\xi)$, $\theta_\rho(\xi)$ of total grade $\partial\xi$; and $i(\xi)$ of total grade $1 + \partial\xi$, $\xi \in L$.

We have, consequently,

$$[\theta_0(\xi), \rho(\eta)] = 0 \tag{3.17}$$

$$[\rho(\xi), \rho(\eta)] - \rho([\xi, \eta]) = \Omega_\rho(\xi, \eta) \tag{3.18}$$

$$[\delta_0, (\rho\wedge)] + (\rho\wedge)^2 = \Omega_\rho \wedge \tag{3.19}$$

$$[(\rho\wedge), i(\xi)] = 0 \tag{3.20}$$

$$[i(\xi), \rho(\eta)] = 0. \tag{3.21}$$

Proof. For (i) and (ii) we refer to [I, 2.2][11].

(iii) A first proof is obtained by adapting the results in [I, 2.3] to the change of definition (1.6) above. A second proof arises from the fact that both sides of each of the above relations (3.4) through (3.21) have the same nature as module derivations (cf. [5.3] below)—while they agree in low grade:

[3.3] Lemma. *Relations* (3.4) *through* (3.15) *hold in restriction to* $\bigwedge^0(L, E)$ *and* $\bigwedge^1(L, E)$.

This lemma is checked using the definition formulae in [3.1] whose low grade restrictions we list below:

[3.4] We have the following formulae for the classical operators in low grade: for $\lambda_0 \in \bigwedge^0(L, E)^{\partial\lambda_0}$, $\lambda_1 \in \bigwedge^1(L, E)^{\partial\lambda_1}$, $\lambda_2 \in \Lambda^2(L, E)^{\partial\lambda_2}$, and $\xi, \eta, \xi_1, \xi_2, \xi_3 \in L^{\cdot}$:

$$\begin{aligned}
(\delta_0\lambda_0)(\xi_1) &= 0\\
(\delta_0\lambda_1)(\xi_1, \xi_2) &= -\lambda_1([\xi_1, \xi_2])\\
(\delta_0\lambda_2)(\xi_1, \xi_2, \xi_3) &= -\lambda_2([\xi_1, \xi_2], \xi_3) + (-1)^{\partial\xi_2\,\partial\xi_3}\lambda_2([\xi_1, \xi_3], \xi_2)\\
&\quad - (-1)^{\partial\xi_1(\partial\xi_2+\partial\xi_3)}\lambda_2([\xi_2, \xi_3], \xi_1)
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
(\rho\wedge\lambda_0)(\xi_1) &= (-1)^{\partial\xi_1\,\partial_0\lambda_0}\rho(\xi_1)\,\lambda_0\\
(\rho\wedge\lambda_1)(\xi_1, \xi_2) &= (-1)^{\partial\xi_1\,\partial_0\lambda_1}\rho(\xi_1)\,\lambda_1(\xi_2) - (-1)^{\partial\xi_2(\partial_0\lambda_1+\partial\xi_1)}\rho(\xi_2)\,\lambda_1(\xi_1)\\
(\rho\wedge\lambda_2)(\xi_1, \xi_2, \xi_3) &= (-1)^{\partial\xi_1\,\partial_0\lambda_2}\rho(\xi_1)\,\lambda_2(\xi_2, \xi_3)\\
&\quad - (-1)^{\partial\xi_2(\partial_0\lambda_2+\partial\xi_1)}\rho(\xi_2)\,\lambda_2(\xi_1, \xi_3)\\
&\quad + (-1)^{\partial\xi_3(\partial_0\lambda_2+\partial\xi_1+\partial\xi_2)}\rho(\xi_3)\,\lambda_2(\xi_1, \xi_2)
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\theta_0(\xi)\,\lambda_0 &= 0\\
\{\theta_0(\xi)\,\lambda_1\}(\xi_1) &= -(-1)^{\partial\xi(1+\partial_0\lambda_1)}\lambda_1([\xi, \xi_1])\\
\{\theta_0(\xi)\,\lambda_2\}(\xi_1, \xi_2) &= -(-1)^{\partial\xi\,\partial_0\lambda_2}\{\lambda_2([\xi, \xi_1], \xi_2) + (-1)^{\partial\xi\,\partial\xi_1}\lambda_2(\xi_1, [\xi, \xi_2])
\end{aligned} \tag{3.24}$$

[11] However, see remark at the end of Section 2 for an independent proof of (i).

$$\begin{aligned}
\rho(\xi)\,\lambda_0 &= \rho(\xi)\,\lambda_0\\
\{\rho(\xi)\,\lambda_1\}(\xi_1) &= (-1)^{\partial\xi}\rho(\xi)\,\lambda_1(\xi_1)\\
\{\rho(\xi)\,\lambda_2\}\{\xi_1, \xi_2) &= \rho(\xi)\,\lambda_2(\xi_1, \xi_2)
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
i(\xi)\,\lambda_0 &= 0\\
i(\xi)\,\lambda_1 &= (-1)^{\partial\xi(1+\partial_0\lambda_1)}\lambda_1(\xi)\\
\{i(\xi)\,\lambda_2\}(\xi_1) &= (-1)^{\partial\xi\,\partial_0\lambda_2}\lambda_2(\xi, \xi_1)
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\Omega_\rho(\xi, \eta)\,\lambda_0 &= \Omega_\rho(\xi, \eta)\,\lambda_0\\
\{\Omega_\rho(\xi, \eta)\,\lambda_1\}(\xi_1) &= (-1)^{\partial\xi+\partial\eta}\Omega_\rho(\xi, \eta)\,\lambda_1(\xi_1)
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
(\Omega_\rho\wedge\lambda_0)(\xi_1, \xi_2) &= (-1)^{\partial_0\lambda_0(\partial\xi_1+\partial\xi_2)}\Omega_\rho(\xi_1, \xi_2)\,\lambda_0\\
(\Omega_\rho\wedge\lambda_1)(\xi_1, \xi_2, \xi_3) &= (-1)^{\partial_0\lambda_1(\partial\xi_1+\partial\xi_2)}\Omega_\rho(\xi_1, \xi_2)\\
&\quad - (-1)^{\partial\xi_2\,\partial\xi_3+\partial_0\lambda_1(\partial\xi_1+\partial\xi_2)}\Omega_\rho(\xi_1, \xi_3)\,\lambda_1(\xi_2)\\
&\quad + (-1)^{(\partial\xi_2+\partial\xi_3)(\partial\xi_1+\partial_0\lambda_1)}\Omega_\rho(\xi_2, \xi_3)\,\lambda_1(\xi_1).
\end{aligned} \tag{3.28}$$

Proof of Lemma [3.3]. Straightforward verifications are immediate for $\bigwedge^0(L, E)$. As for $\bigwedge^1(L, E)$: (3.11) is immediate; (3.12) follows from (3.12a) (which boils down to the graded Jacobi identity), (3.17), and (3.18); (3.13) follows from (3.13a) (again boiling down to the Jacobi identity) and (3.19) (straightforward—6 terms cancelling in the calculation of $[\delta_0, \rho\wedge]\,\lambda_1$); (3.14) follows from (3.14a) (immediate) and (3.20); (3.15) follows from (3.15a) and (3.21) (immediate).

[3.4] *Remark.* The operator $\rho(\xi)$ defined in (3.6) is not a connection in the $\bigwedge^*(L, A)$-module $\bigwedge^*(L, E)$. This can be cured by extending the definition (3.1): given a pair of connections (∇, ρ), with ∇ an L-connection and ρ an E-connection, one defines the operators θ^∇ and ρ_∇ by

$$\begin{aligned}
\{\theta^\nabla(\xi)\,\lambda\}(\xi_1, ..., \xi_n) &= \tfrac{1}{2}(-1)^{n\,\partial\xi+1}\sum_{i=1}^{n}(-1)^{\gamma_i}\\
&\quad \times \{\lambda(\xi_1, ..., \xi_{i-1}, [\xi, \xi_i] + T_\nabla(\xi, \xi_i), \xi_{i+1}, ..., \xi_n)\},\\
&\quad \xi_1, ..., \xi_n \in L, \xi \in L^{\cdot}\\
\rho_\nabla(\xi) &= \theta^\nabla(\xi) + \rho(\xi).
\end{aligned} \tag{3.29}$$

Then ρ_∇ is an $\bigwedge^*(L, E)$-connection which is local if ρ and ∇ are local. In the particular case of $\nabla = \mathrm{ad}$ we obtain $\theta^\nabla = \theta_0$ and $\rho_\nabla = \theta_\rho$ (cf. [2.3]).

4. The Graded Commutative Differential Algebra $\bigwedge^*(L, A)$

We now study the basic A-module, namely A itself, which will then serve to construct the other A-modules E via tensor products (cf. Section [4]). A carries a natural connection d determined by the action of L:

$$d(\xi)\, a = \xi a, \qquad \xi \in L,\ a \in A. \tag{4.1}$$

This connection is flat and local owing to [2.1](i) and (2.3).

[4.1] THEOREM. *Let (L, A) be a graded Lie Cartan pair, with $\delta = \delta_d$, $d\wedge$, δ_0; $\theta(\xi) = \theta_d(\xi)$, $\theta_0(\xi)$, $d(\xi)$; $i(\xi)$, $\xi \in L$ the classical operators attached to the A-connection (4.1).*[12] *We have that*

(i) *$\bigwedge^*(L, A)$ equipped*

— *with the $\mathbb{N}$-grading with restriction n on $\bigwedge^n(L, A)$, the* **intrinsic grading** *determined by*

$$\partial_0 \alpha = \partial[\alpha(\xi_1, ..., \xi_n)] - \sum_{i=1}^{n} \partial \xi_i, \qquad \xi_i \in L^{\cdot} \tag{4.2}$$

and the **total grading** $\partial\alpha = \partial_0\alpha + n$

— *with the bilinear* **wedge product** *specified as follows: for $\alpha \in \bigwedge^p(L, A)^{\partial\alpha}$ and $\beta \in \bigwedge^q(L, A)^{\partial\beta}$ we set*

$$\alpha \wedge \beta = (-1)^{q\,\partial_0\alpha} \frac{(p+q)!}{p!\, q!} A_{p+q}(\alpha \otimes \beta), \tag{4.3}$$

where

$$\begin{gathered}(\alpha \otimes \beta)(\xi_1, ..., \xi_{p+q}) = (-1)^{\partial_0\beta \sum_{i=1}^{p} \partial\xi_i} \alpha(\xi_1, ..., \xi_p)\, \beta(\xi_{p+1}, ..., \xi_{p+q}) \\ \xi_1, ..., \xi_p \in L^{\cdot} \text{ of grades } \partial\xi_1, ..., \partial\xi_p, \text{ resp. } \xi_{p+1}, ..., \xi_{p+q} \in L\end{gathered} \tag{4.4}$$

— *with the* **differential** $\delta = \delta_{\mathrm{d}}$

is a bigraded differential algebra,[13] *graded commutative w.r.t. its total grading, with $\bigwedge^*_A(L, A)$ a differential subalgebra stable under $\theta(\xi)$, δ, and $i(\xi)$ for all $\xi \in L$.*

(ii) *We have, moreover, the following derivation properties: for α, $\beta \in \bigwedge^*(L, A)$, α of total grade $\partial\alpha$; and $\xi \in L^{\partial\xi}$, $\eta \in L^{\partial\eta}$, we have in addition to*

$$\delta(\alpha \wedge \beta) = (\delta\alpha) \wedge \beta + (-1)^{\partial\alpha} \alpha \wedge \delta\beta, \tag{4.5}$$

the properties

[12] Note that since d is flat, one has $\Omega_d(\xi, \eta) = 0$, ξ, $\eta \in L$; and $\Omega_d \wedge = 0$.

[13] See Appendix A for definitions.

$$d \wedge (\alpha \wedge \beta) = (d \wedge \alpha) \wedge \beta + (-1)^{\partial\alpha} \alpha \wedge (d \wedge \beta) \tag{4.6}$$

$$\delta_0(\alpha \wedge \beta) = (\delta_0\alpha) \wedge \beta + (-1)^{\partial\alpha} \alpha \wedge \delta_0\beta \tag{4.7}$$

$$\theta(\xi)(\alpha \wedge \beta) = \{\theta(\xi)\, \alpha\} \wedge \beta + (-1)^{\partial\xi\, \partial\alpha} \alpha \wedge \{\theta(\xi)\, \beta\} \tag{4.8}$$

$$d(\xi)(\alpha \wedge \beta) = \{d(\xi)\, \alpha\} \wedge \beta + (-1)^{\partial\xi\, \partial\alpha} \alpha \wedge \{d(\xi)\, \beta\} \tag{4.9}$$

$$\theta_0(\xi)(\alpha \wedge \beta) = \{\theta_0(\xi)\, \alpha\} \wedge \beta + (-1)^{\partial\xi\, \partial\alpha} \alpha \wedge \{\theta_0(\xi)\, \beta\} \tag{4.10}$$

$$i(\xi)(\alpha \wedge \beta) = \{i(\xi)\, \alpha\} \wedge \beta + (-1)^{(1+\partial\xi)\, \partial\alpha} \alpha \wedge \{i(\xi)\, \beta\}. \tag{4.11}$$

[4.2] COROLLARY. *The operators δ, $d\wedge$, δ_0, $\theta(\xi)$, $\theta_0(\xi)$, $d(\xi)$, $i(\xi)$, fulfill the relations* (3.11) *through* (3.22), *where $\rho = d$, $\Omega_d(\xi, \eta) = 0$, and $\Omega_d \wedge = 0$.*

Proof of the corollary from the theorem. Since both sides of these relations are derivations of the same type, it suffices, to check the agreement in restriction to $\bigwedge^0(L, A)$ and $\bigwedge^1(L, A)$; this is, however, a special case of Lemma [3.3].

Proof of the theorem. Definitions (4.4) and (4.3) evidently specify bilinear products $\otimes$ and $\wedge$. We check that these products are associative. Let α, β be as in (4.3), $\gamma \in \bigwedge^r(L, A)$ and $\xi_{p+1}, ..., \xi_{p+q+n} \in L^{\cdot}$ of respective grades $\partial\xi_{p+1}, ..., \partial\xi_{p+q+r}$. We have, owing to (4.4),

$$\begin{aligned}\{(\alpha \otimes \beta) \otimes \gamma\}&(\xi_1, ..., \xi_{p+q+r}) \\ &= \{\alpha \otimes (\beta \otimes \gamma)\}(\xi_1, ..., \xi_{p+q+r}) \\ &= (-1)^{\partial_0\beta \sum_{i=1}^{p} \partial\xi_i + \partial_0\gamma \sum_{i=1}^{p+q} \partial\xi_i} \alpha(\xi_1, ..., \xi_p)\, \beta(\xi_{p+1}, ..., \xi_{p+q})\, \gamma(\xi_{p+q+1}, ..., \xi_{p+q+r})\end{aligned} \tag{4.12}$$

and, on the other hand, owing to (4.3) and (B.5) in Appendix B[14]

$$\begin{aligned}\alpha \wedge (\beta \wedge \gamma) &= (-1)^{(q+r)\,\partial_0\alpha} \frac{(p+q+r)!}{p!(q+r)!} A_{p+q+r}(\alpha \otimes (\beta \wedge \gamma)) \\ &= (-1)^{(q+r)\,\partial_0\alpha + r\,\partial_0\beta} \frac{(p+q+r)!}{p!(q+r)!} \frac{(q+r)!}{q!\, r!} A_{p+q+r}(\alpha \otimes A_{q+r}(\beta \otimes \gamma)) \\ &= (-1)^{(q+r)\,\partial_0\alpha + r\,\partial_0\beta} \frac{(p+q+r)!}{p!\, q!\, r!} A_{p+q+r}(\alpha \otimes \beta \otimes \gamma) \\ &= (-1)^{r(\partial_0\alpha + \partial_0\beta) + q\,\partial_0\alpha} \frac{(p+q+r)!}{(p+q)!\, r!} \frac{(p+q)!}{p!\, q!} A_{p+q+r}(A_{p+q}(\alpha \otimes \beta) \otimes \gamma) \\ &= (\alpha \wedge \beta) \wedge \gamma.\end{aligned} \tag{4.13}$$

We proved that $\bigwedge^*(L, A)$ is an associative algebra under the wedge product $\wedge$. It is then an obvious consequence of definitions (4.3), (4.4) that $\bigwedge^*(L, A)$ becomes a

[14] See Appendix B of [I] for more details.

bigraded complex algebra with total grading (4.2) and $\mathbb{N}$-grading as above. For the rest of our proof we shall need properties of the wedge product gathered in the next two lemmas.

[4.3] LEMMA. *Let* $\varphi_1, ..., \varphi_{n+1} \in \bigwedge^1 (L, A)$ *of respective intrinsic grades* $\partial_0\varphi_1, ..., \partial_0\varphi_{n+1}$, *and total grades* $\partial\varphi_1, ..., \partial\varphi_n$; *and let* $\sigma \in \Sigma_n$. *We have that*

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n = (-1)^{(n-1)\,\partial_0\varphi_1 + (n-2)\,\partial_0\varphi_2 + \cdots + \partial_0\varphi_{n-1}}\, n!\, A_n(\varphi_1 \otimes \cdots \otimes \varphi_n) \tag{4.14}$$

$$\sigma_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \chi_n(\varphi, \sigma^{-1})\, \varphi_{\sigma^{-1}1}, \otimes \cdots \otimes \varphi_{\sigma^{-1}n} \tag{4.15}$$

$$\varphi_1 \wedge \cdots \wedge \varphi_n = (-1)^{\sum_{i=1}^{n-1} (n-i)\,\partial_0\varphi_i} \sum_{\sigma \in \Sigma_n} \chi_n(\varphi, \sigma)\, \varphi_{\sigma 1} \otimes \cdots \otimes \varphi_{\sigma n} \tag{4.16}$$

(*see* (B, 2) *in Appendix* B *for the definition of* χ_n: $\chi_n(\varphi, \sigma) = \chi(\sigma) \sum_{i>j, \sigma i<\sigma j} \partial_0\varphi_{\sigma i}\, \partial_0\varphi_{\sigma j}$)

$$\varphi_1 \wedge \cdots \wedge \varphi_{n+1} = \sum_{i=1}^{n+1} (-1)^{n\,\partial_0\varphi_i + \partial\varphi_i \sum_{k=1}^{i-1} \partial\varphi_k}\, \varphi_i \otimes \{\varphi_1 \wedge \cdots \wedge \hat{\varphi}_i \wedge \cdots \wedge \varphi_{n+1}\}. \tag{4.17}$$

Proof. (4.14) is a special case of the formula

$$\begin{gathered} \alpha_1 \wedge \cdots \wedge \alpha_n = (-1)^{\sum_{i=1}^{n-1} N_{i+1}\,\partial_0\alpha_i}\, \frac{N_1!}{\prod_{i=1}^{n} p_i!}\, A_{N_1}(\alpha_1 \otimes \cdots \otimes \alpha_n) \\ \alpha_i \in \bigwedge^{p_i} (L, A) \qquad \text{with } N_s = \sum_{i=s}^{n} p_s \end{gathered} \tag{4.18}$$

proven in (4.13) for $n=3$ and obtained from there by induction w.r.t. n.

Proof of (4.15). We have

$$\begin{aligned} &\{\sigma_n(\varphi_1 \otimes \cdots \otimes \varphi_n)\}(\xi_1, ..., \xi_n) \\ &\quad = \chi_n(\xi, \sigma)\{\varphi_1 \otimes \cdots \otimes \varphi_n\}(\xi_{\sigma 1}, ..., \xi_{\sigma n}) \\ &\quad = \chi_n(\xi, \sigma)(-1)^{\sum_{i>j} \partial_0\varphi_i\, \partial\xi_{\sigma j}}\, \varphi_1(\xi_{\sigma 1}) \cdots \varphi_n(\xi_{\sigma n}) \\ &\quad = \chi_n(\xi, \sigma)(-1)^{\sum_{i>j} \partial_0\varphi_i\, \partial\xi_{\sigma j}}\, \chi_n^+(\varphi.(\xi_{\sigma \cdot}), \sigma^{-1}) \\ &\qquad \times \varphi_{\sigma^{-1}1}(\xi_1), ..., \varphi_{\sigma^{-1}n}(\xi_n) \\ &\quad = \chi_n(\xi, \sigma)\, \chi_n^+(\varphi.(\xi_{\sigma \cdot}), \sigma^{-1})(-1)^{A+B}\, \{\varphi_{\sigma^{-1}1} \otimes \cdots \otimes {}_{\sigma^{-1}n}\}(\xi_1, ..., \xi_n), \end{aligned} \tag{4.19}$$

where $A = \sum_{i>j} \partial_0\varphi_i\, \partial\xi_{\sigma j}$, $B = \sum_{i>j} \partial_0\varphi_{\sigma^{-1}i}\, \partial\xi_j$, and

$$\begin{aligned} \chi_n^+(\varphi.(\xi_{\sigma \cdot}, \sigma^{-1}) &= \chi_n^+(\varphi_{\sigma \cdot^{-1}}(\xi.), \sigma) \\ &= (-1)^{\sum_{i>j, \sigma i<\sigma j} (\partial_0\varphi_i + \partial\xi_{\sigma i})(\partial_0\varphi_j + \partial\xi_{\sigma j})} \\ &= \chi_n^+(\xi, \sigma)(-1)^{C+D+E}, \end{aligned} \tag{4.20}$$

where

$$C = \sum_{\substack{i>j \\ \sigma i<\sigma j}} \partial_0\varphi_i\, \partial_0\varphi_j, \qquad D = \sum_{\substack{i>j \\ \sigma i<\sigma j}} \partial_0\varphi_i\, \partial\xi_{\sigma j}, \qquad E = \sum_{\substack{i>j \\ \sigma i<\sigma j}} \partial_0\varphi_j\, \partial\xi_{\sigma i}. \tag{4.21}$$

Now we have

$$\begin{aligned} A + D + E &= \sum_{\substack{i>j \\ \sigma i>\sigma j}} \partial_0\varphi_i\, \partial\xi_{\sigma j} + \sum_{\substack{i<j \\ \sigma i>\sigma j}} \partial_0\varphi_i\, \partial\xi_{\sigma j} \\ &= \sum_{\sigma i>\sigma j} \partial_0\varphi_i\, \partial\xi_{\sigma j} = \sum_{i>j} \partial_0\varphi_{\sigma-1_i}\, \partial\xi_j = B. \end{aligned} \tag{4.22}$$

Therefore the numerical factor in front of the last expression in (4.19) boils down to $\chi(\sigma)(-1)^C$ yielding (4.15), since one has $C = \sum_{i>j, \sigma^{-1}i<\sigma^{-1}j} \partial_0\varphi_{\sigma^{-1}i}\, \partial_0\varphi_{\sigma^{-1}j}$. Relation (4.16) then follows from (4.14) and (4.15). We now check (4.17) as follows: write (4.16) with $n \to n+1$ and decompose Σ_{n+1} as follows:

$$\Sigma_{n+1} = \bigcup_{i=1}^{n+1} \bigcup_{\tau \in \Sigma_n^i} \hat{\tau} \circ \sigma_i \tag{4.23}$$

with σ_i and $\hat{\tau}$ the permutations

$$\sigma_i = \begin{pmatrix} 1, 2, \ldots\ldots\ldots\ldots\ldots, n+1 \\ i, 1, ..., i-1, i+1, ..., n+1 \end{pmatrix} \tag{4.24}$$

$$\hat{\tau} = \begin{pmatrix} i, 1, ..., i-1, i+1, \ldots\ldots\ldots\ldots, n+1 \\ i, \tau 1, ..., \tau(i-1), \tau(i+1), ..., \tau(n+1) \end{pmatrix}, \tag{4.25}$$

τ running through the group Σ_n^i of permutations of $\{1, 2, ..., i-1, i+1, ..., n+1\}$. We have

$$\chi_{n+1}(\varphi, \sigma_i) = (-1)^{i-1 + \partial_0\varphi_i \sum_{k=1}^{i-1} \partial_0\varphi_k} \tag{4.26}$$

$$\chi_{n+1}(\varphi \circ \sigma_i, \sigma_i^{-1}\hat{\tau}\sigma_i) = \chi_n(\varphi_1, ..., \hat{\varphi}_i, ..., \varphi_{n+1}; \tau). \tag{4.27}$$

We thus have, again using (4.16),

$$\begin{aligned} \varphi_1 \wedge \cdots \wedge \varphi_{n+1} &= (-1)^U \sum_{\sigma \in \Sigma_{n+1}} \chi_{n+1}(\varphi, \sigma)\, \varphi_{\sigma 1} \otimes \cdots \otimes \varphi_{\sigma(n+1)} \\ &= (-1)^U \sum_{i=1}^{n+1} \sum_{\tau \in \Sigma_n^i} \chi_{n+1}(\varphi, \hat{\tau} \circ \sigma_i) \\ &\qquad \times \varphi_i \otimes \varphi_{\tau 1} \otimes \cdots \otimes \varphi_{\tau(i-1)} \otimes \varphi_{\tau(i+1)} \otimes \cdots \otimes \varphi_{\tau(n+1)} \\ &= (-1)^U \sum_{i=1}^{n+1} \chi_{n+1}(\varphi, \sigma_i) \sum_{\tau \in \Sigma_n^i} \chi_n(\varphi_1, ..., \hat{\varphi}_i, ..., \varphi_{n+1}; \tau) \\ &\qquad \times \varphi_i \otimes \varphi_{\tau 1} \otimes \cdots \otimes \varphi_{\tau(i-1)} \otimes \varphi_{\tau(i+1)} \otimes \cdots \otimes \varphi_{\tau(n+1)} \\ &= (-1)^U \sum_{i=1}^{n+1} \chi_n(\varphi, \sigma_i)\, \varphi_i \otimes (-1)^V \varphi_1 \wedge \cdots \wedge \hat{\varphi}_i \wedge \cdots \wedge \varphi_{n+1} \end{aligned} \tag{4.28}$$

with

$$\begin{aligned} U &= n\partial_0\varphi_1 + (n-1)\,\partial_0\varphi_2 + \cdots + (n+1-i)\,\partial_0\varphi_i + \cdots + \partial_0\varphi_n \\ V &= (n-1)\,\partial_0\varphi_1 + \cdots + (n-i+1)\,\partial_0\varphi_{i-1} + (n-i)\,\partial_0\varphi_{i+1} + \cdots + \partial_0\varphi_{n+1}; \end{aligned} \tag{4.29}$$

hence

$$U+V = \sum_{k=1}^{i-1} \partial_0\varphi_k + (n-i+1)\,\partial_0\varphi_i; \tag{4.30}$$

thus

$$(-1)^{U+V}\chi_{n+1}(\varphi,\sigma_i) = (-1)^{\partial\varphi \sum_{k=1}^{i-1}\partial\varphi_k + n\,\partial_0\varphi_i}; \tag{4.31}$$

whence (4.17).

[4.4] LEMMA. *Let $a\in A=\bigwedge^0(L,A)$, $\varphi\in\bigwedge^1(L,A)$, and $\psi\in\bigwedge^2(L,A)$ of respective intrinsic grades ∂a, $\partial_0\varphi$, and $\partial_0\psi$. And let $\beta\in\bigwedge^n(L,A)$. We have, for $\xi_1,\dots,\xi_{n+2}\in L$ of grades $\partial\xi_1,\dots$, resp. $\partial\xi_{n+2}$:*

$$(a\wedge\beta)(\xi_1,\dots,\xi_n) = (-1)^{n\,\partial a}\,a\beta(\xi_1,\dots,\xi_n) \tag{4.32}$$

$$\begin{aligned}(\varphi\wedge\beta)(\xi_1,\dots,\xi_{n+1}) &= (-1)^{n\,\partial_0\varphi}\sum_{i=1}^{n+1}(-1)^{1+i+\partial\xi_i(\partial_0\beta+\sum_{k=1}^{i-1}\partial\xi_i)} \\ &\quad\times\varphi(\xi_i)\,\beta(\xi_1,\dots,\hat{\xi}_i,\dots,\xi_{n+1})\end{aligned} \tag{4.33}$$

$$\begin{aligned}(\psi\wedge\beta)(\xi_1,\dots,\xi_{n+2}) &= -(-1)^{n\,\partial_0\psi}\sum_{1\leqslant i<j\leqslant n+2}(-1)^{\alpha_{ij}+\partial_0\beta(\partial\xi_i+\partial\xi_j)} \\ &\quad\times\psi(\xi_i,\xi_j)\,\beta(\xi_1,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_{n+2})\end{aligned} \tag{4.34}$$

where the caret ^ indicates a missing argument and α_{ij} is as in (3.2).

Proof. Straightforward specializations of the definition (4.4): (4.32) is immediate; (4.33), resp. (4.34), follows by writing $\Sigma_{n+1}=\bigcup_{i=1}^n\bigcup_{\sigma\in\Sigma_n^i}\hat{\sigma}\circ\sigma_i$, resp. $\Sigma_{n+2}=\bigcup_{i=1}^{n+2}\bigcup_{\tau\in\Sigma_n^{ij}}\hat{\hat{\tau}}\circ\sigma_{ij}$, σ_i, σ_{ij}, $\hat{\sigma}$, $\hat{\hat{\tau}}$ the permutations

$$\begin{aligned}
&\sigma_i=\begin{pmatrix}1, 2, \dots\dots\dots, n+1\\ i, 1, \dots, \hat{\imath}, \dots, n+1\end{pmatrix}, \qquad \chi_{n+1}(\xi,\sigma_i)=(-1)^{1+i+\partial\xi_i\sum_{k=1}^{i-1}\partial\xi_k}\\
&\sigma_{ij}=\begin{pmatrix}1, 2, 3, \dots\dots\dots\dots, n+2\\ i, j, 1, \dots, \hat{\imath}, \dots, \hat{\jmath}, \dots, n+2\end{pmatrix}, \qquad \chi_{n+2}(\xi,\sigma_{ij})=-(-1)\,\alpha_{ij}\\
&\hat{\sigma}=\begin{pmatrix}i, 1, \dots, i-1, \dots, i+1, \dots\dots\dots, n+1\\ i, \sigma 1, \dots, \sigma(i-1), \sigma(i+1), \dots, \sigma(n+1)\end{pmatrix}\\
&\hat{\hat{\tau}}=\begin{pmatrix}i, j, 1, \dots, i-1, i+1, \dots, j-1, j+1, \dots\dots\dots\dots\dots\dots, n+2\\ i, j, \tau 1, \dots, \tau(i-1)\,\tau(i+1), \dots, \tau(j-1), \tau(j+1), \dots, \tau(n+2)\end{pmatrix}
\end{aligned} \tag{4.35}$$

(Observe that $\chi_{n+1}(\xi\circ\sigma_i,\ \sigma_i^{-1}\hat{\sigma}\sigma_i)=\chi_n(\xi^i,\sigma)$, $\chi_{n+2}(\xi\circ\sigma_{ij},\sigma_{ij}^{-1}\hat{\hat{\tau}}\sigma_{ij})=\chi_n(\xi^{ij},\tau)$, where $\xi^i=(\xi_1,\dots,\hat{\xi}_i,\dots,\xi_{n+1})$, $\xi^{ij}=(\xi_1,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_{n+2})$.)

End of Proof of Theorem [4.1]. We shall perform the remaining proof replacing $\bigwedge^*(L,A)$ by the subalgebra[15] $\bigwedge^*_{\mathrm{DR}}(L,A)$ linearly generated by totally decomposable tensors (= generated by $\bigwedge^1(L,A)$ as an algebra). The results, in fact, hold for $\bigwedge^*(L,A)$ but on $\bigwedge^*_{\mathrm{DR}}(L,A)$ one gets shorter and more instructive proofs using induction arguments.[16]

We first check the graded commutativity of $\bigwedge^*(L,A)$, whereby it suffices to consider one-forms φ_1, $\varphi_2\in\bigwedge^1(L,A)$ of respective intrinsic (resp. total) grades $\partial_0\varphi_1$, $\partial_0\varphi_2$ (resp. $\partial\varphi_1$, $\partial\varphi_2$): we have from (4.17)

$$\begin{aligned}\varphi_1\wedge\varphi_2 &= (-1)^{\partial_0\varphi_1}\{\varphi_1\otimes\varphi_2-(-1)^{\partial_0\varphi_1\,\partial_0\varphi_2}\,\varphi_2\otimes\varphi_1\}\\ &= -(-1)^{\partial_0\varphi_1\,\partial_0\varphi_2+\partial_0\varphi_1+\partial_0\varphi_2}\\ &\quad\times(-1)^{\partial_0\varphi_2}\{\varphi_2\otimes\varphi_1-(-1)^{\partial_0\varphi_1\,\partial_0\varphi_2}\,\varphi_1\otimes\varphi_2\}\\ &= (-1)^{\partial\varphi_1\,\partial\varphi_2}\,\varphi_2\wedge\varphi_1.\end{aligned} \tag{4.36}$$

We now prove the derivation properties (4.5) through (4.11). In view of Lemma [A.1] in Appendix A, it is enough to check these properties for the first factor α of the wedge product a zero- and a one-form: let thus $a\in A=\bigwedge^0(L,A)$ with $\partial a=\partial_0 a$; $\varphi\in\bigwedge^1(L,A)$ with $\partial\varphi=1+\partial_0\varphi$; and $\beta\in\bigwedge^n(L,A)$ with $\partial\beta=\partial_0\beta+n$.

Proof of (4.7). We have, from (3.2) and (4.32),

$$\begin{aligned}\delta_0(a\wedge\beta) &= (-1)^{n\,\partial a}\,\delta_0(a\beta)=(-1)^{n\,\partial a}\,a\,\delta_0\beta\\ &= (-1)^{n\,\partial a+(n+1)\,\partial a}\,a\wedge\delta_0\beta=(-1)^{\partial a}\,a\wedge\delta_0\beta\\ &= \delta a\wedge\beta+(-1)^{\partial a}\,a\wedge\delta_0\beta.\end{aligned} \tag{4.37}$$

On the other hand, it follows from (4.4) and (3.2), using (4.33) and (4.34), that we have

$$(\varphi\otimes\beta)(\xi_1,\dots,\xi_{n+1})=(-1)^{\partial\xi_1\,\partial_0\beta}\,\varphi(\xi_1)\,\beta(\xi_2,\dots,\xi_{n+1}) \tag{4.38}$$

and, since $(\delta_0,\xi_2)=-\varphi([\xi_1,\xi_2])$ (cf. (3.22)),

$$\begin{aligned}\{\delta_0(\varphi\otimes\beta)\}(\xi_1,\dots,\xi_{n+1}) &= -\sum_{1\leqslant i<j\leqslant n+2}(-1)^{(\partial\xi_i+\partial\xi_j)\,\partial_0\beta+\alpha_{ij}}\\ &\quad\times(\delta_0\varphi)(\xi_i,\xi_j)\,\beta(\xi_1,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_{n+1})\\ &= (-1)^{n\,\partial_0\varphi}\{\delta_0\varphi\wedge\beta\}(\xi_1,\dots,\xi_{n+1}),\end{aligned} \tag{4.39}$$

[15] DR stands for De Rham (since one obtains in that way the classical De Rham complex in the case $A=C^\infty(M)$). Note that one has in general $\bigwedge^*_{\mathrm{DR}}(L,A)\subset\bigwedge^*_A(L,A)$.

[16] See below for an alternative general proof. Note that $\bigwedge^*_A(L,A)$ is generated by $\bigwedge^1_A(L,A)$ as an algebra if L is a finite projective A-module.

we found that

$$\delta_0(\varphi \otimes \beta) = (-1)^{n\,\partial_0\varphi}\,(\delta_0\varphi) \wedge \beta. \tag{4.40}$$

Applying this to the r.h.s. of (4.17) yields

$$\begin{aligned}\delta_0(\varphi_1 \wedge \cdots \wedge \varphi_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{\partial\varphi_i \sum_{k=1}^{i-1} \partial\varphi_k}\,(\delta\varphi_i) \wedge \varphi_1 \wedge \cdots \wedge \hat{\varphi}_i \wedge \cdots \wedge \varphi_{n+1} \\ &= \sum_{i=1}^{n} (-1)^{\sum_{k=1}^{i-1}\partial\varphi_k}\,\varphi_1 \wedge \cdots \wedge \delta\varphi_i \wedge \cdots \wedge \varphi_{n+1},\end{aligned} \tag{4.41}$$

where we used graded commutativity. Setting $\varphi = \varphi_1$ and $\beta = \varphi_2 \wedge \cdots \wedge \varphi_{n+1}$, this yields an inductive proof of (4.7) for $\alpha = \varphi$.

Proof of (4.5) *and* (4.6). Since $\delta = \delta_d = \delta_0 + d\wedge$, (4.5) follows from (4.6) and (4.7). As for the latter, we have, from (3.3) on the one hand, using (4.32), (4.33),

$$\begin{aligned}&d \wedge (a \wedge \beta)(\xi_1, ..., \xi_{n+1}) \\ &\quad= (-1)^{n\,\partial a} \sum_{i=1}^{n+1} (-1)^{1+i+\partial\xi_i(\partial a + \partial_0\beta + \sum_{k=1}^{i-1}\partial\xi_k)} \\ &\qquad\times \{(\xi_i a)\,\beta(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1}) + (-1)^{\partial a\,\partial\xi_i}\,a\xi_i(\beta(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1}))\} \\ &\quad= \{(d \wedge a) \wedge \beta + (-1)^{\partial a}\,a \wedge (d \wedge \beta)\}(\xi_1, ..., \xi_{n+1}).\end{aligned} \tag{4.42}$$

On the other hand, for the calculation of $d \wedge (\varphi \wedge \beta)$ we note the analogy of formula (4.33) and formula (3.3)[17] which reads, for $\rho = d$

$$\begin{aligned}(d \wedge \lambda)(\xi_1, ..., \xi_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{1-i+\partial\xi_i(\partial_0\lambda \sum_{k=1}^{i-1}\partial\xi_k)} \\ &\qquad\times d(\xi_i)\,\lambda(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1})\end{aligned} \tag{4.43}$$

We have to calculate (4.43) for $\lambda = \varphi \wedge \beta$ given by (4.33); we get a sum of expressions of the type

$$d(\xi_i)[\varphi(\xi_j)\,\beta(\xi_1, ..., \hat{\xi}_j, ..., \xi_{n+1})]_{\hat{\xi}_i}, \tag{4.44}$$

where $[\;]_{\hat{\xi}_i}$ indicates the shift of variables $\xi_{i+k} \to \xi_{i+k+1}$, $k = 1, ..., n+1-i$. Because of the derivation property of $d(\xi_i)$, (4.44) is a sum of two terms where $d(\xi_i)$ acts on the first, resp. second factor of the product $[\;]_{\hat{\xi}_i}$. Now owing to the analogy noted above and to associativity and graded commutativity of the wedge product, the first summands will add up to $(d \wedge \varphi) \wedge \beta$, and the second to $(-1)^{\partial\varphi}\,\varphi \wedge (d \wedge \beta)$.

[17] This analogy motivates the notation $\rho\wedge$ (note that ρ is of intrinsic grade 0 (cf. Definition [2.1]).

Proof of (4.10). We first study the action of $\theta_0(\xi)$ on a tensor product; we have, from (4.4) and (3.5), for $\alpha \in \bigwedge^p (L, A)_{\partial_0\alpha}$, $\beta \in \bigwedge^q (L, A)_{\partial_0\beta}$:

$$\begin{aligned}&\{\theta_0(\xi)(\alpha \otimes \beta)\}(\xi_1, ..., \xi_{p+q}) \\ &\quad= (-1)^{\partial_0\beta \sum_{k=1}^{p}\partial\xi_k + (p+q)\,\partial\xi + 1} \\ &\qquad\times \Big\{\sum_{i=1}^{p} (-1)^{\partial\xi(\partial_0\alpha + \sum_{k=1}^{i-1}\partial\xi_k)}\,\alpha(\xi_1, ..., [\xi, \xi_i], ..., \xi_p)\,\beta(\xi_{p+1}, ..., \xi_{p+q}) \\ &\qquad+ \sum_{i=1}^{q} (-1)^{\partial\xi(\partial_0\alpha + \partial_0\beta + \sum_{k=1}^{p}\partial\xi_k + \sum_{k=p+1}^{p+i-1}\partial\xi_k)} \\ &\qquad\times \alpha(\xi_1, ..., \xi_p)\,\beta(\xi_{p+1}, ..., [\xi, \xi_{p+i}], ..., \xi_{p+q})\Big\} \\ &\quad= (-1)^{q\,\partial\xi}\,(-1)^{\partial_0\beta \sum_{k=1}^{p}\partial\xi_k}\,\{\theta_0(\xi)\,\alpha\}(\xi_1, ..., \xi_p)\,\beta(\xi_{p+1}, ..., \xi_{p+q}) \\ &\qquad+ (-1)^{\partial\alpha\,\partial\xi + (\partial_0\beta + \partial\xi)\sum_{k=1}^{p}\partial\xi_k}\,\alpha(\xi_1, ..., \xi_p)\{\theta_0(\xi)\,\beta\}(\xi_{p+1}, ..., \xi_{p+q}) \\ &\quad= \{(-1)^{q\,\partial\xi}\,\theta_0(\xi)\,\alpha \otimes \beta + (-1)^{\partial\alpha\,\partial\xi}\,\alpha \otimes \theta_0(\xi)\}(\xi_1, ..., \xi_{p+1}).\end{aligned} \tag{4.45}$$

We proved the property

$$\begin{gathered}\theta_0(\xi)(\alpha \otimes \beta) = (-1)^{q\,\partial\xi}\,\theta_0(\xi)\,\alpha \otimes \beta + (-1)^{\partial\alpha\,\partial\xi}\,\alpha \otimes \theta_0(\xi)\,\beta, \\ \alpha \in \textstyle\bigwedge^p (L, A)^{\partial\alpha}, \qquad \beta \in \textstyle\bigwedge^q (L, A), \qquad \xi \in L^{\partial\xi}.\end{gathered} \tag{4.46}$$

On the other hand, we have that $\theta_0(\xi)$ commutes with all permutations, thus with A_n:

$$\begin{gathered}\theta_0(\xi)\,\sigma_n = \sigma_n\theta_0(\xi), \qquad \sigma \in \Sigma_n,\ \xi \in L, \\ \theta_0(\xi)\,A_n = A_n\theta_0(\xi).\end{gathered} \tag{4.47}$$

Indeed $\theta_0(\xi)$ commutes with all transpositions τ_k: $k \Leftrightarrow k+1$, $1 \leqslant k \leqslant n-1$: denoting by $\theta_0^i(\xi)\,\lambda$ the ith term in the r.h.s. of (3.5) we have namely

$$\begin{aligned}\theta_0^i(\xi)(\tau_k\lambda) &= \tau_k\{\theta_0^i(\xi)\,\lambda\}, \qquad k < i,\ k > i+1 \\ \theta_0^k(\xi)(\tau_k\lambda) &= \tau_k\{\theta_0^{k+1}(\xi)\,\lambda\} \\ \theta_0^{k+1}(\xi)(\tau_k\lambda) &= \tau_k\{\theta_0^k(\xi)\,\lambda\}.\end{aligned} \tag{4.48}$$

Commuting A_{p+q} and $\theta_0(\xi)$, we then have from (4.46)

$$\begin{aligned}\theta_0(\xi)(\alpha \wedge \beta) &= (-1)^{q\,\partial_0\alpha}\,\frac{(p+q)!}{p!\,q!}\,A_{p+q}\{(-1)^{q\,\partial\xi}\,\theta_0(\xi)\,\alpha \otimes \beta + (-1)^{\partial\alpha\,\partial\xi}\,\alpha \otimes \theta_0(\xi)\,\beta\} \\ &= \theta_0(\xi)\,\alpha \wedge \beta + (-1)^{\partial\xi\,\partial\alpha}\,\alpha \wedge \theta_0(\xi)\,\beta.\end{aligned} \tag{4.49}$$

Proof of (4.8) *and* (4.9). (4.8) follows from (4.10) and (4.9). For the latter, we proceed as previously. We have, for a tensor product, from (4.4) and (3.6)

$$\begin{aligned}\{d(\xi)(\alpha\otimes\beta)\}(\xi_1,\ldots,\xi_p)&\\ &=(-1)^{(p+q)\,\partial\xi+\partial_0\beta\,\Sigma_{k=1}^{p}\partial\xi_k}\,d(\xi)\{\alpha(\xi_1,\ldots,\xi_p)\,\beta(\xi_{p+1},\ldots,\xi_{p+q})\}\\ &=(-1)^{(p+q)\,\partial\xi+\partial_0\beta\,\Sigma_{k=1}^{p}\partial\xi_k}\,\{\xi\{\alpha(\xi_1,\ldots,\xi_p)\}\cdot\beta(\xi_{p+1},\ldots,\xi_{p+q})\\ &\quad+(-1)^{\partial\xi(\partial_0\alpha+\Sigma_{k=1}^{p}\partial\xi_k)}\,\alpha(\xi_1,\ldots,\xi_p)\,\xi[\beta(\xi_{p+1},\ldots,\xi_{p+q})]\},\end{aligned}\tag{4.50}$$

showing that we have

$$\begin{gathered}d(\xi)(\alpha\otimes\beta)=(-1)^{q\,\partial\xi}\,d(\xi)\,\alpha\otimes\beta+(-1)^{\partial\alpha\,\partial\xi}\,\alpha\otimes d(\xi)\,\beta,\\ \alpha\in\textstyle\bigwedge^{p}(L,A)^{\partial\alpha},\qquad \beta\in\textstyle\bigwedge^{q}(L,A),\qquad \xi\in L^{\partial\xi},\end{gathered}\tag{4.51}$$

from which (4.9) is proven as above, since $d(\xi)$ evidently commutes with σ_n, $\sigma\in\Sigma_n$.

Proof of (4.11). For $\alpha=\varphi\in\bigwedge^1(L,A)_{\partial_0\varphi}$ and $\beta\in\bigwedge^n(L,A)_{\partial_0\beta}$, we note that this amounts to the commutation relation

$$[i(\xi),\varphi\wedge\,]=i(\xi)(\varphi\wedge\,)-(-1)^{(1+\partial\xi)\,\partial\varphi}\,(\varphi\wedge\,)\,i(\xi)=(-1)^{\partial\varphi\,\partial\xi}\,\varphi(\xi)\wedge.\tag{4.52}$$

To check the latter, we write (4.33), isolating the first term:

$$\begin{aligned}(\varphi\wedge\beta)(\xi_1,\ldots,\xi_{n+1})&\\ &=(-1)^{n\,\partial_0\varphi+\partial\xi_1\,\partial_0\beta}\,\varphi(\xi_1)\,\alpha(\xi_2,\ldots,\xi_{n+1})\\ &\quad+(-1)^{n\,\partial_0\varphi}\sum_{k=2}^{n+1}(-1)^{1+k+\partial\xi_k(\partial_0\beta+\Sigma_{j=1}^{k-1}\partial\xi_j)}\,\varphi(\xi_k)\,\beta(\xi_1,\ldots,\hat{\xi}_k,\ldots,\xi_{n+1}).\end{aligned}\tag{4.53}$$

We then have from (3.7), (4.32), (4.33),

$$\begin{aligned}\{i(\xi)(\varphi\wedge\beta)\}(\xi_1,\ldots,\xi_n)&\\ &=(-1)^{(n+1+\partial_0\varphi)\,\partial\xi+n\,\partial_0\varphi}\,\varphi(\xi)\,\beta(\xi_1,\ldots,\xi_n)\\ &\quad+(-1)^{(n+1+\partial_0\varphi+\partial_0\beta)\,\partial\xi+n\,\partial_0\varphi}\sum_{k=2}^{n+1}(-1)^{1+k+\partial\xi_{k-1}(\partial_0\beta+\partial\xi+\Sigma_{i=1}^{k-2}\partial\xi_i)}\\ &\quad\times\varphi(\xi_{k-1})\,\beta(\xi,\xi_1,\ldots,\hat{\xi}_{k-1},\ldots,\xi_n)\\ &=(-1)^{\partial\varphi\,\partial\xi}\,\{\varphi(\xi)\wedge\beta\}(\xi_1,\ldots,\xi_n)\\ &\quad+(-1)^{(1+\partial\xi)\partial\varphi}\,\{\varphi\wedge i(\xi)\,\beta\}(\xi_1,\ldots,\xi_n).\end{aligned}\tag{4.54}$$

We have finished the proof of the derivation properties [4.1](ii).[18] We conclude this section with three remarks.

First, our proof of these derivation properties entails a proof of the fact that δ, $d\wedge$, δ_0, $\theta(\xi)$, $\theta_0(\xi)$, $d(\xi)$ leave $\bigwedge^*(L,A)$ invariant. Indeed, we could have defined these operators as the corresponding derivations with restrictions to $\bigwedge^0(L,A)$ and $\bigwedge^1(L,A)$ specified in (3.4) for $\rho=d$ (cf. Lemma [A.1]) (it is immediate that these restrictions act within these spaces). The calculations in our proofs would then show that this definition amounts to the specification by the formulae (3.5) through (3.7) with $\rho=d$.

The present proofs make it intuitive that the derivation properties could be checked directly from the definition formulae, and thus hold without the assumption that $\bigwedge^*(L,A)$ is spanned by decomposable tensors. In fact we gave a direct proof of (4.8), (4.9), (4.10). A direct verification of (4.11) is cumbersome but practicable. For (4.7), direct verification is extremely cumbersome, but a general proof is obtained by the following detour: one can directly verify the "Cartan relation" (3.14) which, in combination with (4.8) then allows an inductive proof (cf. [I]).

We conclude with a remark about the relationship between the "classical differential forms" in $\bigwedge^*(L,A)$ and Connes' generalized differential forms in $\Omega(A)$ (cf. [7], or [9] for the present $\mathbb{Z}/2$-graded case). Since $\Omega(A)$ is universal, and $\bigwedge^0(L,A)=A$, we have by Corollary [1.9] in [9] the commutative diagram

$$\begin{tikzcd}\Omega(A) \arrow[drr] & & \\ A \arrow[u, hook] \arrow[rr, hook] & & \bigwedge^*(L,A)\end{tikzcd}\tag{4.55}$$

where the oblique arrow is a homomorphism of bigraded differential algebras (onto $\bigwedge^*_{\mathrm{DR}}(L,A)$); hence the classical differential forms are homomorphic images of elements of $\Omega(A)$ (cf. [8, 9, Appendix E]).

5. The Classical Operators Attached to an E-Connection as Module Derivations

In this section we take advantage of the isomorphism $\bigwedge^*(L,E)=E\otimes A\bigwedge^*(L,A)$ to give a description of the classical operators attached to an E-connection ρ in terms of the classical operators on $\bigwedge^*(L,A)$ attached to the connection d described in the former section. This, together with the derivation property of the latter, implies module-derivation properties, which in turn imply Theorem [3.2](iii) in conjunction with Lemma [3.3].

[18] These properties could also be adapted from the corresponding results in [I].

[5.1] LEMMA. *With (L, A) a graded Lie–Cartan pair, and E a graded unital finite projective right A-module,*[19] *the convention*

$$(X \otimes \alpha)(\xi_1, ..., \xi_n) = (-1)^{n\,\partial X} X\alpha(\xi_1, ..., \xi_n), \qquad X \in E^{\cdot}\ \alpha \in \bigwedge^n (L, A) \tag{5.1}$$

establishes grade-zero isomorphisms

$$E \otimes_A \bigwedge^* (L, A) \simeq \bigwedge^* (L, E) \tag{5.2}$$

and

$$E \otimes_A \bigwedge_A^* (L, A) \simeq \bigwedge_A^* (L, E), \tag{5.3}$$

where the tensor products in the l.h.s. of (5.2) and (5.3) are effected via the left A-module structure of $\bigwedge^* (L, A)$, resp. $\bigwedge_A^* (L, A)$, stemming from the identification $\bigwedge^0 (L, A) = \bigwedge_A^0 (L, A) = A$,[20] *i.e.,*

$$a\alpha = a \wedge \alpha, \qquad \begin{cases} a \in A = \bigwedge^0 (L, A) \\ \alpha \in \bigwedge^* (L, A)\ (\text{resp. } \bigwedge_A (L, A)). \end{cases} \tag{5.4}$$

The set $\bigwedge^* (L, E)$ (resp. $\bigwedge_A^* (L, E)$) then becomes a graded unital finite projective right $\bigwedge^* (L, A)$-module (resp. $\bigwedge_A^*$-module), with

$$\begin{gathered} (X \otimes \alpha)\, \beta = X \otimes (\alpha \wedge \beta) \\ X \in E, \qquad \alpha, \beta \in \bigwedge^* (L, A) \qquad (\text{resp.} \in \bigwedge_A^* (L, A)). \end{gathered} \tag{5.5}$$

Note that the above identifications commute with the $\mathbb{N}$-grading and the intrinsic grading (hence the total grading) in the sense that

$$\begin{gathered} E \otimes A \bigwedge^n (L, A) \simeq \bigwedge^n (L, E), \qquad n \in \mathbb{N} \\ E^k \otimes A \bigwedge^n (L, A)^p = \bigwedge^n (L, E)^{p+k}, \qquad p, k \in \mathbb{Z}/2. \end{gathered} \tag{5.6}$$

Proof. It is clear that (5.1) establishes a linear map $E \otimes \bigwedge^* (L, A) \subset \bigwedge^* (L, E)$, for $\otimes$ the tensor product over $\mathbb{C}$. Moreover, since it gives, for $a \in A^{\cdot}$

$$\begin{aligned} (Xa \otimes \alpha)(\xi_1, ..., \xi_n) &= (-1)^{n(\partial X + \partial a)} Xa\alpha(\xi_1, ..., \xi_n) \\ &= (-1)^{n\,\partial X} X\{(a \wedge \alpha)(\xi_1, ..., \xi_n)\} \end{aligned} \tag{5.7}$$

[19] If we drop the finite projective assumption, we have isomorphic inclusions instead of isomorphisms.

[20] Note that this identification entails the identification $E = E \otimes_A \bigwedge^0 (L, A)$ such that $X = X \otimes \mathbb{1}$, $X \in E$, in agreement with our former identifications $E = \bigwedge^0 (L, E)$.

it follows from (5.4) that $E \otimes_A \bigwedge^* (L, A) \subsetneqq \bigwedge^* (L, E)$. However, with (e_i, ε^i) a coordinatization of the finite projective A-module E, $(e_i \otimes \mathbb{1}, \varepsilon^i \otimes \mathbb{1})$ is a coordinatization of the $\bigwedge^* (L, A)$-module $E \otimes_A \bigwedge^* (L, A)$, for which the $e_i \otimes \bigwedge^* (L, A)$ span $\bigwedge^* (L, E)$: the above inclusion is thus surjective. Moreover, the isomorphism (5.2) follows from the fact that we have, for $\alpha \in \bigwedge_A^n (L, E)$, $a \in A^{\cdot}$ $\xi_1, ..., \xi_k \in L^{\cdot}$:

$$\begin{aligned} (X \otimes \alpha)(\xi_1, ..., a\xi_i, ..., \xi_n) &= (-1)^{n\,\partial X} X\alpha(\xi_1, ..., a\xi_i, ..., \xi_n) \\ &= (-1)^{n\,\partial X + \partial a \sum_{k=i+1}^n \partial \xi_k} X\alpha(\xi_1, ..., \xi_n)\, a \\ &= (-1)^{\partial a \sum_{k=i+1}^n \partial \xi_k} \{(X \otimes \alpha)(\xi_1, ..., \xi_n)\}\, a. \end{aligned} \tag{5.8}$$

The next lemma now expresses the classical operators attached to an E-connection ρ in terms of those attached to the A-connection d.

[5.2] LEMMA. *Let (L, A) be a graded Lie–Cartan pair, with ρ an E-connection. We then have, for $X \in E^{\cdot}$ and $\alpha \in \bigwedge^* (L, A)$, $\xi, \eta \in L^{\cdot}$, with $X = X \otimes \mathbb{1}$ (cf. footnote 20)*

$$\delta_\rho(X \otimes \alpha) = (\delta_\rho X) + (-1)^{\partial X} X \otimes \delta\alpha \tag{5.9}$$

$$\rho \wedge (X \otimes \alpha) = (\rho \wedge X)\, \alpha + (-1)^{\partial X} X \otimes (d \wedge \alpha) \tag{5.10}$$

$$\delta_0(X \otimes \alpha) = (-1)^{\partial X} X \otimes \delta_0 \alpha \tag{5.11}$$

$$\theta_\rho(\xi)(X \otimes \alpha) = \{\theta_\rho(\xi)\, X\}\, \alpha + (-1)^{\partial \xi\, \partial X} X \otimes \{\theta(\xi)\, \alpha\} \tag{5.12}$$

$$\theta_0(\xi)(X \otimes \alpha) = (-1)^{\partial \xi\, \partial X} X \otimes \{\theta_0(\xi)\, \alpha\} \tag{5.13}$$

$$\rho(\xi)(X \otimes \alpha) = \{\rho(\xi)\, X\}\, \alpha + (-1)^{\partial \xi\, \partial X} X \otimes \{d(\xi)\, \alpha\} \tag{5.14}$$

$$i(\xi)(X \otimes \alpha) = (-1)^{(1 + \partial \xi)\, \partial X} X \otimes \{i(\xi)\, \alpha\} \tag{5.15}$$

$$\Omega_\rho(\xi, \eta)(X \otimes \alpha) = \{\Omega_\rho(\xi, \eta)\, X\}\, \alpha \tag{5.16}$$

$$\Omega_\rho \wedge (X \otimes \alpha) = (\Omega_\rho \wedge X)\, \alpha. \tag{5.17}$$

For the proof, we need

[5.3] LEMMA. *For $Z \in \bigwedge^1 (L, E)^{\partial Z}$ and $\alpha \in \bigwedge^n (L, A)$ we have, $\hat{}$ indicating a missing argument:*

$$\begin{aligned} &(Z\alpha)(\xi_1, ..., \xi_{n+1}) \\ &= (-1)^{n\,\partial_0 Z} \sum_{i=1}^{n+1} (-1)^{1 + i + \partial \xi_i (\partial_0 \alpha + \sum_{k=1}^{i-1} \partial \xi_k)} Z(\xi_i)\, \alpha(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1}). \end{aligned} \tag{5.18}$$

Proof. Immediate from (5.1), (5.4) and (4.3), (4.4).

Proof of Lemma [5.2]. Let $\alpha \in \bigwedge^n (L, A)\ \partial_0 \alpha$ and $\xi_1, ..., \xi_{n+2} \in L^{\cdot}$; (5.9) follows from (5.10) and (5.11). Check of (5.10): we have, using (3.3), (5.18), and (2.5), with $\beta_i = 1 + i + \partial \xi_i (\partial_0 \alpha + \sum_{k=1}^{i-1} \partial \xi_k)$

$$\{\rho \wedge (X \otimes \alpha)\}(\xi_1, ..., \xi_{n+1})$$

$$= \sum_{i=1}^{n+1} (-1)^{\beta_i + \partial\xi_i \partial X} \rho(\xi_i)\{(-1)^{n\,\partial X} X\alpha(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1})\}$$

$$= (-1)^{n\,\partial X} \sum_{i=1}^{n+1} (-1)^{\beta_i + \partial\xi_i \partial X} \{\rho(\xi_i) X\} \alpha(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1})\}$$

$$+ (-1)^{n\,\partial X} \sum_{i=1}^{n+1} (-1)^{\beta_i} \xi_i\{\alpha(\xi_1, ..., \hat{\xi}_i, ..., \xi_{n+1}\}$$

$$= \{(\rho \wedge X)\alpha + (-1)^{\partial X} X \otimes (d \wedge \alpha)\}(\xi_1, ..., \xi_{n+1}). \tag{5.19}$$

Check of (5.11): using (3.2), with the notation α_{ij} there,

$$\{\delta_0(X \otimes \alpha)\}(\xi_1, ..., \xi_{n+1})$$

$$= \sum_{0 \leq i < j \leq n+1} (-1)^{\alpha_{ij} + n(\partial X)} X\alpha([\xi_i, \xi_j], \xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_{n+1})$$

$$= (-1)^{n(\partial X)} X\{(\delta_0\alpha)(\xi_1, ..., \xi_{n+1})\}$$

$$= (-1)^{\partial X} X \otimes \delta_0\alpha. \tag{5.20}$$

Next, (5.12) follows from (5.13) and (5.14). Check of (5.12): we have, from (3.5), with $\gamma_i = \partial\xi(\partial_0\alpha + \sum_{k=1}^{i-1} \partial\xi_k)$,

$$\{\theta_0(\xi)(X \otimes \alpha)\}(\xi_1, ..., \xi_n)$$

$$= (-1)^{n\,\partial\xi + 1 + n\,\partial X} \sum_{i=1}^{n} (-1)^{\gamma_i + \partial\xi\,\partial X} X\alpha(\xi_1, ..., \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, ..., \xi_n)$$

$$= (-1)^{\partial\xi\,\partial X + n\,\partial X + n\,\partial\xi + 1} X \sum_{i=1}^{n} (-1)^{\gamma_i} \alpha(\xi_1, ..., \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, ..., \xi_n)$$

$$= (-1)^{\partial\xi\,\partial X + n\,\partial X} X\{\theta_0(\xi)\alpha\}(\xi_1, ..., \xi_n)$$

$$= (-1)^{\partial\xi\,\partial X} \{X \otimes \theta_0(\xi)\alpha\}(\xi_1, ..., \xi_n). \tag{5.21}$$

Check of (5.14): we have from (3.6), using (2.5),

$$\{\rho(\xi)(X \otimes \alpha)\}(\xi_1, ..., \xi_n) = (-1)^{n\,\partial\xi + n\,\partial X} \rho(\xi)\{X\alpha(\xi_1, ..., \xi_n)\}$$

$$= (-1)^{n(\partial\xi + \partial X)} \{\rho(\xi) X\} \alpha(\xi_1, ..., \xi_n)$$

$$+ (-1)^{\partial\xi\,\partial X + n\,\partial X} X(-1)^{n\,\partial\xi} \xi\{\alpha(\xi_1, ..., \xi_n)\}$$

$$= \{(\rho(\xi) X) \otimes \alpha + (-1)^{\partial\xi\,\partial X} X \otimes \{d(\xi)\alpha\}\}(\xi_1, ..., \xi_n). \tag{5.22}$$

Check of (5.15): we have, from (3.7),

$$\{i(\xi)(X \otimes \alpha)\}(\xi_1, ..., \xi_{n-1}) = (-1)^{(n + \partial_0\alpha + \partial X)\,\partial\xi + n\,\partial X} X\alpha(\xi, \xi_1, ..., \xi_{n-1})$$

$$= (-1)^{(\partial\xi + 1)\,\partial X + (n-1)\,\partial X} X(-1)^{n + \partial_0\alpha} \alpha(\xi, \xi_1, ..., \xi_{n-1})$$

$$= (-1)^{(\partial\xi + 1)\,\partial X} \{X \otimes i(\xi)\alpha\}(\xi_1, ..., \xi_{n-1}). \tag{5.23}$$

Check of (5.15): we have, from (3.8) and [2.2](iii),

$$\{\Omega_\rho(\xi, \eta)(X \otimes \alpha)(\xi_1, ..., \xi_n) = (-1)^{n(\partial\xi + \partial\eta) + n\,\partial X} \Omega_\rho(\xi, \eta)\{X\alpha(\xi_1, ..., \xi_n)\}$$

$$= (-1)^{n(\partial\xi + \partial\eta + \partial X)} \{\Omega_\rho(\xi, \eta) X\} \alpha(\xi_1, ..., \xi_n)$$

$$= \{\Omega_\rho(\xi, \eta) X\} \otimes \alpha. \tag{5.24}$$

Check of (5.16): we have, from (3.9) and [2.2](iii), with α_{ij} as in (3.2),

$$\{\Omega_\rho \wedge (X \otimes \alpha)\}(\xi_1, ..., \xi_{n+2})$$

$$= - \sum_{1 \leq i < j \leq n+2} (-1)^{\alpha_{ij} + (\partial X + \partial_0\alpha)(\partial\xi_i + \partial\xi_j) + n\,\partial X} \Omega_\rho(\xi_i, \xi_j)\{X\alpha(\xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_{n+2})\}$$

$$= \{(\Omega_\rho X)\alpha\}(\xi_1, ..., \xi_{n+2}). \tag{5.25}$$

[5.3] THEOREM. *Let $[L, A]$ be a graded Lie–Cartan pair with ρ an E-connection. The corresponding classical operators have the following properties: for $\xi, \eta \in L$ $\lambda \in \bigwedge^* (L, E) = E \otimes_A \bigwedge^* (L, A)$ of total grade $\partial\lambda$, and $\alpha \in \bigwedge^* (L, A)$, we have*

$$\delta_\rho(\lambda\alpha) = (\delta_\rho\lambda)\alpha + (-1)^{\partial\lambda} \lambda(\delta\alpha) \tag{5.26}$$

$$\rho \wedge (\lambda\alpha) = (\rho \wedge \lambda)\alpha + (-1)^{\partial\lambda} \lambda(d \wedge \alpha) \tag{5.27}$$

$$\delta_0 \wedge (\lambda\alpha) = (\delta_0\lambda)\alpha + (-1)^{\partial\lambda} \lambda(\delta_0\alpha) \tag{5.28}$$

$$\theta_\rho(\xi)(\lambda\alpha) = \{\theta_\rho(\xi)\lambda\}\alpha + (-1)^{\partial\xi\,\partial\lambda} \lambda\{\theta(\xi)\alpha\} \tag{5.29}$$

$$\theta_0(\xi)(\lambda\alpha) = \{\theta_0(\xi)\lambda\}\alpha + (-1)^{\partial\xi\,\partial\lambda} \lambda\{\theta_0(\xi)\alpha\} \tag{5.30}$$

$$\rho(\xi)(\lambda\alpha) = \{\rho(\xi)\lambda\}\alpha + (-1)^{\partial\xi\,\partial\lambda} \lambda\{d(\xi)\alpha\} \tag{5.31}$$

$$i(\xi)(\lambda\alpha) = \{i(\xi)\lambda\}\alpha + (-1)^{(1 + \partial\xi)\,\partial\lambda} \lambda\{i(\xi)\alpha\} \tag{5.32}$$

$$\Omega_\rho(\xi, \eta)(\lambda\alpha) = \{\Omega_\rho(\xi, \eta)\lambda\}\alpha \tag{5.33}$$

$$\Omega_\rho \wedge (\lambda\alpha) = (\Omega_\rho \wedge \lambda)\alpha. \tag{5.34}$$

Proof. It is enough to prove these relations for $\lambda = X \otimes \beta$, $X \in E$ of grade ∂X, $\beta \in \bigwedge^* (L, A)$ of total grade $\partial\beta$, this allowing us to take advantage of the relations in [5.2], combined with the derivation properties (4.5) through (4.12).

Check of (5.26): we have

$$\begin{aligned}\delta_\rho((X\otimes\beta)\,\alpha) &= \delta_\rho(X\otimes(\alpha\wedge\beta))\\ &= (\delta_\rho X)\otimes\alpha\wedge\beta+(-1)^{\partial X}X\otimes\{(\delta\alpha)\wedge\beta+(-1)^{\partial\alpha}\alpha\wedge\delta\beta\}\\ &= \{(\delta_\rho X)\,\alpha+(-1)^{\partial X}X\otimes\delta\alpha\}\,\beta+(-1)^{\partial X+\partial\alpha}X\otimes(\alpha\wedge\delta\beta)\\ &= \{\delta_\rho(X\otimes\alpha)\}\,\beta+(-1)^{\partial(X\otimes\alpha)}(X\otimes\alpha)\,\delta\beta. \end{aligned} \tag{5.35}$$

The proofs of (5.27), (5.28) are identical, modulo the changes $\delta_\rho\to\rho$, $\delta\to d\wedge$, resp. $\delta_\rho\to\delta_0$, $\delta\to\delta_0$.

Check of (5.29): we have

$$\begin{aligned}&\theta_\rho(\xi)\{(X\otimes\alpha)\,\beta\}\\ &\quad=\theta_\rho(\xi)(X\otimes(\alpha\wedge\beta))\\ &\quad=\{\theta_\rho(\xi)\,X\}(\alpha\wedge\beta)+(-1)^{\partial\xi\,\partial X}X\otimes[\{\theta_\rho(\xi)\,\alpha\}\wedge\beta+(-1)^{\partial\xi\,\partial\alpha}\alpha\wedge\theta_\rho(\xi)\,\beta]\\ &\quad=[\{\theta_\rho(\xi)\,X\}\,\alpha+(-1)^{\partial\xi\,\partial X}X\otimes\theta_\rho(\xi)\,\alpha]\,\beta+(-1)^{\partial\xi(\partial X+\partial\alpha)}X\otimes(\alpha\wedge\theta_\rho(\xi)\,\beta)\\ &\quad=\{\theta_\rho(\xi)(X\otimes\alpha)\}\,\beta+(-1)^{\partial\xi\,\partial(X\otimes\alpha)}(X\otimes\alpha)\,\theta_\rho(\xi)\,\beta. \end{aligned} \tag{5.36}$$

The proofs of (5.30), (5.31) are identical modulo the changes $\theta_\rho(\xi)\to\theta_0(\xi)$, $\theta(\xi)\to\theta_0(\xi)$, resp. $\theta_\rho(\xi)\to\rho(\xi)$, $\theta(\xi)\to d(\xi)$.

Check of (5.32): we have

$$\begin{aligned}&i(\xi)\{(X\otimes\alpha)\,\beta\}\\ &\quad=i(\xi)\{X\otimes(\alpha\wedge\beta)\}\\ &\quad=\{i(\xi)\,X\}(\alpha\wedge\beta)+(-1)^{(1+\partial\xi)\,\partial X}X\otimes[\{i(\xi)\,\alpha\}\wedge\beta+(-1)^{(1+\partial\xi)\,\partial\alpha}\alpha\wedge i(\xi)\,\beta]\\ &\quad=[\{i(\xi)\,X\}\,\alpha+(-1)^{(1+\partial\xi)\,X}X\otimes i(\xi)\,\alpha]\,\beta+(-1)^{(1+\partial\xi)(\partial X+\partial\alpha)}X\otimes(\alpha\wedge i(\xi)\,\beta)\\ &\quad=\{i()(X\otimes\alpha)\}\,\beta+(-1)^{(1+\partial\xi)\,\partial(X\otimes\alpha)}(X\otimes\alpha)\,i(\xi)\,\beta. \end{aligned} \tag{5.37}$$

Check of (5.33):

$$\begin{aligned}\Omega_\rho(\xi,\eta)\{(X\otimes\alpha)\,\beta\} &= \Omega_\rho(\xi,\eta)(X\otimes(\alpha\wedge\beta))\\ &= \{\Omega_\rho(\xi,\eta)\,X\}(\alpha\wedge\beta)=[\{\Omega_\rho(\xi,\eta)\,X\}\,\alpha]\,\beta\\ &= [\Omega_\rho(\xi,\eta)(X\otimes\alpha)]\,\beta. \end{aligned} \tag{5.38}$$

The proof of (5.39) is identical modulo the change $\Omega_\rho(\xi,\eta)\to\Omega_\rho\wedge$.

End of proof of Theorem [3.2]. Relation [3.11]: (5.32) says that $i(\xi)$ is an $i(\xi)$-derivation of the $\bigwedge^*(L,A)$-module $\bigwedge^*(L,E)$. Hence $[i(\xi),i(\eta)]$ is a $[i(\xi),i(\eta)]$-derivation.[21] Since the latter vanishes in grades 0 and 1 (cf. [3.3]), it vanishes throughout. Relation (3.13): δ_ρ is a δ-derivation by (5.6), hence $\delta_\rho^2=\frac{1}{2}[\delta_\rho,\delta_\rho]$ is a $[\delta,\delta]=0$-derivation. But so is $\Omega\wedge$ by (5.34) and the two agree in grades 0 and 1. (3.13a) follows then from (3.13) by passing to the depletion, and (3.19) by difference.[22]

Relation (3.14). $[\delta_\rho, i(\xi)]$ is a $[\delta, i(\xi)]=\theta(\xi)$-derivation (cf. [4.2]), and so is $\theta_\rho(\xi)$; and these agree in grades 0 and 1. (3.14a) then follows from (3.14) by passing to the depletion.

Relation (3.15). $[i(\xi),\theta_\rho(\eta)]$ is a $[i(\xi),\theta(\eta)]=i([\xi,\eta])$-derivation (cf. [4.2]) and so is $i([\xi,\eta])$; and these agree in grades 0 and 1. (3.15a) then follows by passing to the depletion.

Proof of (3.16). We have, from (3.13) and (3.14)

$$\begin{aligned}[i(\xi),\Omega_\rho\wedge] &= [i(\xi),\delta_\rho^2]=[i(\xi),\delta_\rho]\,\delta_\rho-(-1)^{1+\partial\xi}\,\delta_\rho[i(\xi),\delta_\rho]\\ &= (-1)^{\partial\xi}\,\theta_\rho(\xi)\,\delta_\rho-\delta_\rho\theta_\rho(\xi)=-[\delta_\rho,\theta_\rho(\xi)]. \end{aligned} \tag{5.39}$$

APPENDIX A. Graded Algebras, Graded Modules, and Derivations

A $\mathbb{Z}/2$-*graded complex*[23] *vector space* is a complex vector space E with a direct decomposition $E\equiv E^0\oplus E^1$ (or equivalently with a *grading involution*, i.e., a linear operator ε of square $\mathbb{1}$ which determines E^0 and E^1 as its eigenspaces with eigenvalue $+1$, resp. -1). The elements of E^0, resp. E^1, are called *even*, resp. *odd*, *vectors*; their *grade* is by definition 0 mod 2, resp. 1 mod 2. The set $E^0\cup E^1$ of *homogeneous* elements of E will be denoted $E^\cdot$.

A $\mathbb{Z}/2$-*graded complex algebra* is a graded complex vector space $\mathscr{A}=\mathscr{A}^0\oplus\mathscr{A}^1$ with a bilinear product

$$\mathscr{A}\times\mathscr{A}\ni(a,b)\to ab\in\mathscr{A} \tag{A.1}$$

such that

$$\mathscr{A}^i\mathscr{A}^j\subset\mathscr{A}^{i+j},\qquad i,j\in\mathbb{Z}/2. \tag{A.2}$$

A $\mathbb{Z}/2$-graded complex algebra $A=A^0\oplus A^1$ is *associative* whenever one has

$$(ab)\,c=a(bc),\qquad a,b,c\in A. \tag{A.3}$$

[21] Cf. Appendix B.

[22] Also results from the agreement in grades 0 and 1 of derivations of the same type.

[23] The specification "complex" could be replaced throughout by "real," and will be omitted whenever clear from the context.

A *complex Lie superalgebra* L is a $\mathbb{Z}/2$-graded complex algebra $L = L^0 \oplus L^1$ whose product, called the *bracket* and denoted $[\ ,\ ]$, fulfills

$$\begin{gathered}[\eta, \xi] = -(-1)^{pq} [\eta, \xi] \\ (-1)^{pr} [\xi, [\eta, \theta]] + (-1)^{qp} [\eta, [\theta, \xi]] + (-1)^{rq} [\theta, [\xi, \eta]] = 0\end{gathered} \tag{A.4}$$

for $\xi, \eta, \theta \in L^{\cdot}$ of respective grades p, q, r.

An associative $\mathbb{Z}/2$-graded complex algebra A becomes a Lie superalgebra under the *graded commutator*, bilinear extension of

$$[a, b] = ab - (-1)^{pq} ba, \qquad a \in A^p, b \in A^q. \tag{A.5}$$

Let E be a $\mathbb{Z}/2$-graded complex vector space with grading involution ε. The set End E of linear operators of E, equipped with the grading involution ad $\varepsilon = \varepsilon \cdot \varepsilon$, is a $\mathbb{Z}/2$-graded complex associative algebra under the operator product—hence a complex Lie superalgebra under the graded commutator of operators.

With A a $\mathbb{Z}/2$-graded complex associative algebra the *derivations* of A are the linear operators D of A fulfilling

$$\begin{aligned} D^0(ab) &= (D^0 a)\, b + a(D^0 b), && a \in A^p,\ b \in A, \\ D^1(ab) &= (D^1 a)\, b + (-1)^p\, a\, D^1 b, && a \in A^p,\ b \in A. \end{aligned} \tag{A.6}$$

Their set Der A is a sub- Lie superalgebra of the complex Lie superalgebra (End A, $[\ ,\ ]$).

Let $A = A^0 \oplus A^1$ be a $\mathbb{Z}/2$-graded complex associative algebra. A *linear graded left* (resp. *right*) *A-module* is a complex $\mathbb{Z}/2$-graded vector space E with a bilinear product

$$\begin{gathered} A \times E \ni (a, X) \to aX \in E \\ (\text{resp. } E \times A \ni (X, a) \to Xa \in E) \end{gathered} \tag{A.7}$$

such that

$$\begin{gathered} A^i E^j \subset E^{i+j} \\ (\text{resp. } E^i A^j \subset E^{i+j}), \quad i, j \in \mathbb{Z}/2 \end{gathered} \tag{A.8}$$

and

$$\begin{gathered} a(bX) = (ab)\, X \\ (\text{resp. } (Xa)\, b = X(ab), \quad a, b \in A, X \in E. \end{gathered} \tag{A.9}$$

With $\delta \in (\text{Der } A)^p$, a *$\delta$-derivation* of E is then a linear operator D of E of grade p fulfilling

$$\begin{aligned} D(aX) &= (\delta a)\, X + (-1)^{pi}\, a\, DX, && a \in A^i,\ X \in E^k, \\ (\text{resp. } D(Xa) &= (DX)\, a + (-1)^{pk}\, X\, \delta a, && a \in A^i,\ X \in E^k. \end{aligned} \tag{A.10}$$

(Note that the 0-derivations of E are the homomorphisms of E.) With D a δ-derivation, and D' a δ'-derivation, of E, $[D, D']$ is a $[\delta, \delta']$-derivation of E. Thus, in particular, the square D^2 of a δ-derivation with δ odd of vanishing square is a homomorphism.

With A and B $\mathbb{Z}/2$-graded complex associative algebras, a *graded left A-, right B-bimodule* is a $\mathbb{Z}/2$-graded complex vector space which is both a graded left A-module and graded right A-module with the additional property

$$(aX)\, b = a(Xb), \qquad X \in E,\ a \in A,\ b \in B \tag{A.11}$$

(for $B = A$, E is called an *A-bimodule*).

With A a $\mathbb{Z}/2$-graded complex associative algebra, E a graded right A-module, and F a graded left A-module, the *tensor product $E \otimes_A F$ of E and F over A* is the quotient of the tensor product $E \otimes F$ of the complex vector spaces E, F by the linear subspace spanned by the elements $Xa \otimes Y - X \otimes aY$, $a \in A$, $X \in E$, $Y \in F$.

If E is a graded left B-, right A-bimodule, and F is a graded left A-right C-bimodule, B, C $\mathbb{Z}/2$-graded associative algebras, $E \otimes_A B$ is then a graded left B-right C-bimodule with the rules

$$\begin{aligned} b(X \otimes Y) &= (bX) \otimes Y, && b \in B,\ c \in C,\ X \in E,\ Y \in F, \\ (X \otimes Y)\, c &= X \otimes (Yc), && b \in B,\ c \in C,\ X \in E,\ Y \in F. \end{aligned} \tag{A.12}$$

A $\mathbb{Z}/2$-graded complex algebra $A = A^0 \oplus A^1$ is *graded-commutative* if associative and such that

$$ba = (-1)^{pq}\, ab, \qquad a \in A^p,\ b \in A^q. \tag{A.13}$$

Graded commutative algebras A are in many ways analogous to commutative algebras (to which they reduce in the case of a *trivial grading*, i.e., $A^0 = A$, $A^1 = \{0\}$). For instance, for A graded commutative, each (graded linear) left A-module E is turned into a (graded linear) right A-module (and reciprocally) by the convention

$$Xa = (-1)^{\partial X\, \partial a}\, aX \tag{A.14}$$

for $X \in E$ of grade ∂X and $a \in A$ of grade ∂a. We thus identify the concept of (graded linear) left (or right) A-module with that of (graded linear) A-bimodule fulfilling (A.14), referring to these objects simply as *A-modules*.[24] In particular, the tensor product $E \otimes_A F$ of two (linear) A-modules E and F over a graded commutative

[24] The notion of δ-derivation is then the same for the left and for the right module structure.

algebra A has the structure of an A-module, this holding also for the tensorial powers $E^{\otimes_A n}$ of E, with the ensuing property

$$\begin{aligned} X_1 \otimes \cdots \otimes X_{i-1} \otimes aX_i \otimes X_{i+1} \otimes \cdots \otimes X_n \\ = (-1)^{\partial a \sum_{k=1}^{i-1} \partial X_k} a(X_1 \otimes \cdots \otimes X_n) \\ = (-1)^{\partial a \sum_{k=i}^{n} \partial X_k} (X_1 \otimes \cdots \otimes X_n)\, a \\ X_i \in E^{\cdot} \qquad a \in A^{\cdot}. \end{aligned} \tag{A.15}$$

The space $\mathrm{Hom}(E^{\otimes_A n}, F)$, F a graded linear A-module, then identifies via the convention

$$\lambda(X_1, ..., X_n) = \lambda(X_1 \otimes \cdots \otimes X_n), \qquad X_i \in E, \tag{A.16}$$

with the graded space $\mathscr{L}^n_A(E, F)$ of *F-valued $n-A$-linear forms* on E, consisting of the n-linear forms $\lambda = \lambda^0 \oplus \lambda^1$ on E with values in F and grading

$$\partial_0 \lambda = \partial\lambda(\xi_1, ..., \xi_n) - \sum_{k=1}^{n} \partial \xi_k, \qquad \xi_1, ..., \xi_k \in E^{\cdot} \tag{A.17}$$

satisfying

$$\begin{aligned} \lambda(X_1, ..., X_{i-1}, aX_i, X_{i+1}, ..., X_n) &= (-1)^{\partial a(\partial_0 \lambda + \sum_{k=i}^{n-1} \partial X_k)} a\lambda(X_1, ..., X_n) \\ &= (-1)^{\partial a \sum_{k=i}^{n} \partial X_k} \lambda(X_1, ..., X_n)\, a \\ X_i \in E^{\cdot} \qquad a \in A^{\cdot}. \end{aligned} \tag{A.18}$$

A further similarity of graded commutative algebras and abelian algebras is the fact that, for A graded commutative, and $\xi \in \mathrm{Der}\, A$, $a \in A$, $a\xi$ defined by $(a\xi)\, n = a(\xi n)$, $n \in A$ again belongs to $\mathrm{Der}\, A$, which thus becomes a left A-module, hence making $(\mathrm{Der}\, A, A)$ a graded Lie–Cartan pair.

A *bigraded complex algebra* is a $\mathbb{Z}/2$-graded complex algebra $\Omega = \Omega^+ \oplus \Omega^-$ (with even part Ω^+ and odd part Ω^-) equipped in addition with a decomposition

$$\Omega = \bigoplus_{n \in \mathbb{N}} \Omega^n \tag{A.19}$$

such that

$$\Omega^n \cdot \Omega^m \subset \Omega^{n+m}, \qquad n, m \in \mathbb{N} \tag{A.20}$$

and

$$\Omega^n = \Omega^{n+} \oplus \Omega^{n-}, \qquad \text{where } \Omega^{n\pm} = \Omega^n \cap \Omega^{\pm},\ n \in \mathbb{N}. \tag{A.21}$$

By definition, the *total grading* $\partial\omega$ *of* $\omega \in \Omega^{n+}$ (resp. $\omega \in \Omega^{n-}$) is 0 mod 2 (resp. 1 mod 2), its $\mathbb{N}$-*grading* is n, and its *intrinsic grading* is $\partial_0 \omega = \partial\omega - n$ mod 2. (Ω, δ) is a *bigraded differential algebra* if Ω is as above and possesses a *differential*. δ, i.e., a derivation of the $\mathbb{Z}/2$-graded algebra $\Omega = \Omega^+ \oplus \Omega^-$ with $\mathbb{N}$-grade and total grade 1:

$$\delta\Omega^n \subset \Omega^{n+1}, \qquad n \in \mathbb{N}, \qquad \delta\Omega^{\pm} \subset \Omega^{\mp}, \tag{A.22}$$

and square zero: $\delta^2 = 0$. The special case of trivial intrinsic grading, $\partial_0 \omega = 0$ (i.e., $\partial\omega = n$ mod 2, $\omega \in \Omega^n$) corresponds to $\mathbb{N}$-*graded* (*differential*) *complex algebras*.

We conclude with a characterization of the derivations of the bigraded associative algebras for which $\Omega - \Omega^0$ is generated by Ω^1.

[A.1] LEMMA. *Let $\Omega = \Omega^+ \oplus \Omega^- = \bigoplus_{n \in \mathbb{N}} \Omega^n$ be a bigraded associative algebra such that Ω^n is "universally spanned" by $(\Omega^1)^n$, $n \geq 1$.*[25]

(i) *For the linear map $D: \Omega \to \Omega$ of total grade p to be a derivation of Ω it suffices that it fulfill*

$$D(a\alpha) = (Da)\, \alpha + (-1)^{p\, \partial a}\, a\, D\alpha \tag{A.23}$$

and

$$D(\varphi\alpha) = (D\varphi)\, \alpha + (-1)^{p\, \partial\varphi}\, \varphi\, D\alpha \tag{A.24}$$

for all $a \in \Omega^0$ of total grade ∂a, all $\varphi \in \Omega^1$ of total grade $\partial\varphi$, and all $\alpha \in \Omega^n$, $n \in \mathbb{N}$.

(ii) *Let $D_0: \Omega^0 \to \Omega$ and $D_1: \Omega^1 \to \Omega$ be linear maps of total grade p fulfilling*

$$\begin{aligned} D_0(ab) &= (D_0 a)\, b + (-1)^{p\, \partial a} D_0 b \\ D_1(a\varphi) &= (D_0 a)\, \varphi + (-1)^{p\, \partial a} D\varphi. \end{aligned} \tag{A.25}$$

Then there is a unique derivation D of Ω (w.r.t. the total grading) restricting to D_0 on Ω^0 and to D_1 on Ω^1. Specifically, one has

$$D(\varphi_1, ..., \varphi_n) = \sum_{i=1}^{n} (-1)^{p \sum_{k=1}^{i-1}} \varphi_1, ..., \varphi_{i-1} (D\varphi_i)\, \varphi_{i+1} \cdots \varphi_n. \tag{A.26}$$

Proof. (i) Assuming that the derivation property holds for α, β:

$$D(\alpha\beta) = (D\alpha)\, \beta + (-1)^{p\, \partial a}\, \alpha\, D\beta, \tag{A.27}$$

we have, for a and φ as in (A.2, 2), on the one hand,

$$\begin{aligned} D(a\alpha\beta) &= (Da)\, \alpha\beta + (-1)^{\partial a}\, a\, D(\alpha\beta) \\ &= (Da)\, \alpha\beta + (-1)^{\partial a}\, a(D\alpha)\, \beta + (-1)^{\partial a + \partial\alpha}\, a\alpha\, D\beta \\ &= D(a\alpha)\, \beta + (-1)^{\partial(a\alpha)}\, (a\alpha)\, D\beta \end{aligned} \tag{A.28}$$

[25] In the sense that n-linear maps on Ω^1 with appropriate symmetry (graded alternate in the case $\Omega = \bigwedge^* (L, A)$) extend to linear maps on Ω^n.

and, analogously,

$$\begin{aligned} D(\varphi\alpha\beta) &= (D\varphi)\,\alpha\beta + (-1)^{\partial\varphi}\,\varphi\,D(\alpha\beta) \\ &= D\varphi\alpha\beta + (-1)^{\partial\varphi}\,\varphi(D\alpha)\,\beta + (-1)^{\partial\alpha+\partial\varphi}\,\varphi\alpha\,D\beta \\ &= D(\varphi\alpha)\,\beta + (-1)^{\partial(\varphi\alpha)}\,\varphi a\,D\beta. \end{aligned} \tag{A.29}$$

Thus (A.27) holds for $a\alpha$, β and for $\varphi\alpha$, β. Hence it holds for arbitrary β and $\alpha = a\varphi_1 \cdots \varphi_p$, $a \in \Omega_0$, $\varphi_1, ..., \varphi_p \in \Omega^1$, i.e., α arbitrary.

(ii) *Uniqueness.* D restricts to D_0 on Ω^0 and acts as in (A.2) on Ω^n by repeated application of the derivation rule.

Existence. Guaranteed by (A.26) which defines D coherently as a linear operator (cf. Footnote 25) and implies both (A.24) and (A.28).

APPENDIX B. Graded Alternate Forms

Let $A = A^0 \oplus A^1$ be a $\mathbb{Z}/2$-graded complex algebra, with $F = F^0 \oplus F^1$ and $E = E^0 \oplus E^1$ two $\mathbb{Z}/2$-graded vector spaces, and denote by $\mathscr{L}^n(F, E)$ the complex vector space of E-valued n-linear[26] forms on F. With $\sigma \in \Sigma_n$, Σ_n the group of permutations of the n first integers, we define σ_n acting on $\mathscr{L}^n(F, E)$ by the relation

$$\begin{aligned} &(\sigma_n\lambda)(\xi_1, ..., \xi_n) = \chi_n(\xi, \sigma)\,\lambda(\xi_{\sigma 1}, ..., \xi_{\sigma n}) \\ &\lambda \in \mathscr{L}^n(F, E), \qquad \{\xi_1, ..., \xi_n\} = \xi \in (F^{\cdot})^n \end{aligned} \tag{B.1}$$

with $\chi_n(\xi, \sigma) = \chi(\sigma)\,\chi_n^+(\xi, \sigma)$, $\chi(\sigma)$ the signature of σ, and[27]

$$\chi_n^+(\xi, \sigma) = (-1)^{\sum_{i>j,\,\sigma i<\sigma j} \partial\xi_{\sigma i}\,\partial\xi_{\sigma j}} = (-1)^{\sum_{i>j,\,\sigma^{-1}i<\sigma^{-1}j} \partial\xi_i\,\partial\xi_j}. \tag{B.2}$$

The $\chi_n(\xi, \sigma)$ are groupoid characters in the sense

$$\chi_n(\xi, \sigma\tau) = \chi_n(\xi\sigma, \tau)\,\chi_n(\xi, \sigma), \qquad \sigma, \tau \in \Sigma_n,\ \xi \in (F^{\cdot})^n, \tag{B.3}$$

where $(\xi\sigma)_i = \xi_{\sigma i}$; furthermore they "split tensorially":

$$\chi_n(\xi \times \eta, \sigma \times \tau) = \chi_n(\xi, \sigma)\,\chi_m(\eta, \tau), \qquad \sigma \in \Sigma_n, \tau \in \Sigma_m,\ \xi \in (F^{\cdot})^n,\ \eta \in (F^{\cdot})^m. \tag{B.4}$$

We define the *graded alternator* as

$$A_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_n. \tag{B.5}$$

With these definitions, we then have that

[26] n-linear means $n - \mathbb{C}$-linear.

[27] $\chi_n^+(\xi, \sigma)$ is thus the sign obtained by combining all the minus signs arising from transpositions of odd elements.

(i) $\sigma \in \Sigma_n \to \sigma_n$ is a linear representation of the group Σ_n on $\mathscr{L}^n(F, E)$ (one has $\sigma_n \cdot \tau_n = (\sigma\tau)_n$ and $(\sigma_n)^{-1} = (\sigma^{-1})_n$) with the properties[28]

$$\sigma_n A_n = A_n \sigma_n = A_n, \qquad \sigma \in \Sigma_n, \tag{B.6}$$

entailing that A_n is an idempotent

$$A_n^2 = A_n. \tag{B.7}$$

The range of A_n consists of the common fixed points of all σ_n, $\sigma \in \Sigma$, called the *E-valued graded-alternate n-linear forms on F*, whose set is denoted $\bigwedge^n (F, E)$:

$$\bigwedge^n (F, E) = A^n \mathscr{L}^n(F, E). \tag{B.8}$$

Note that $\lambda \in \mathscr{L}^n(F, E)$ belongs to $\bigwedge^n (F, E)$ iff

$$\lambda(\xi_{\sigma 1}, ..., \xi_{\sigma n}) = \chi_n(\xi, \sigma)\,\lambda(\xi_1, ..., \xi_n), \qquad \xi_1, ..., \xi_n \in F^{\cdot}. \tag{B.9}$$

(ii) Assuming A graded commutative, and E, F to be A-modules, the subset $\mathscr{L}_A^n(F, E)$ of $\mathscr{L}^n(F, E)$ consisting of $n - A$-linear forms (cf. A, 18) is left invariant by all operators σ_n, $\sigma \in \Sigma_n$: Σ_n thus acts on $\mathscr{L}_A^n(F, E)$ with fixed points the F-valued graded-alternate $n - A$-linear forms on F, whose set we denote $\bigwedge_A^n (F, E)$.

Acknowledgments

The second named author thanks Professor M. Berger for the hospitality of the I.H.E.S., where part of the work was completed. We are grateful to Mrs. Vendla Meyer for a flawless typing of the manuscript. We express our heartfelt thanks to R. Coquereaux for suggesting a number of improvements, and to the referee for pointing out numerous misprints and obscurities.

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[28] Due to the fact that the $\chi(\xi, \sigma)$ (as well as the $\chi^+(\xi, \sigma)$) are groupoid characters in the sense that $\chi(\xi, \sigma\tau) = \chi(\xi, \sigma)\,\chi(\xi\sigma, \sigma\tau)$ for all ξ, σ, and τ.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium