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Harmonic expansion and dimensional reduction in G/H Kaluza-Klein theories

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Abstract. We propose a geometrical framework for harmonic expansion and dimensional reduction of matter fields in Kaluza-Klein theories with the most general G-invariant ansatz. Generalised Peter-Weyl and Frobenius theorems provide a basis for harmonic expansion, and a mechanism is shown by which the dimensional reduction of matter fields is then automatically accomplished. In particular, we discuss the dimensional reduction of tensor and spinor fields, and of the Laplace and Dirac operators.

1. Introduction

Harmonic expansion and dimensional reduction are two mechanisms used in discussing the effective four-dimensional content of higher-dimensional field theories admitting a spontaneous compactification. A general discussion of these problems was given by Salam and Strathdee [1] while Witten [2] gave a broad discussion of the harmonic expansion and dimensional reduction of spinor fields and, in particular, of the difficulties in obtaining the effective four-dimensional chiral asymmetry. Spacetime manifold was treated in those papers locally and no intrinsic geometrical meaning was given to the constructions. On the other hand, certain global aspects of the problem were studied by Römer [3] and Bleecker [4, 4a]; however, both authors restricted their discussion to the case of internal space being a group manifold, and spinors were not discussed at all in [4, 4a]. Manton [5] considered dimensional reduction of fermions interacting with a Yang-Mills field. His discussion was, however, restricted to *invariant* spinors (corresponding to α being the trivial representation of G according to the notation of § 2) and *product* metric.

In this paper we consider a G/H Kaluza-Klein theory as formulated in [6]. As a geometrical model for the extended spacetime we take a manifold E on which a global symmetry group G acts (G can be thought of as the internal symmetry group of a ground state, but the scheme is broad enough to apply to any Riemannian manifold with a compact group of isometries; it will also apply to the still more general case of G being a *local* isometry group, as discussed in [6a]). Then E locally looks like a product $M \times S$ of spacetime M and internal homogeneous space $S \simeq G/H$. The spacetime manifold is defined globally and unambiguously as the manifold of G orbits (internal spaces). What is not unique is a local product representation of E as $M \times G/H$—any two such representations differ by a gauge transformation with gauge

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group N(H)|H (see [6, 7]). All the analysis given in this paper is directed by the aim not to distinguish any particular product representation of E (a global one may even not exist), and in doing so to exhibit an intrinsic geometrical meaning of the constructions.

The outline of the paper is as follows: we start with matter fields on E considered as sections of some equivariant vector bundle over E, i.e. of a vector bundle $\mathscr{F} = \mathscr{F}(E; F)$ with base E and typical fibre F, with G acting on $\mathscr{F}$ by bundle automorphisms. The set $\Gamma\mathscr{F}$ of all cross sections of $\mathscr{F}$ is the space of matter-field configurations in the extended spacetime E. We decompose $\Gamma\mathscr{F}$ according to the irreducible representations α of G and, for each α, define harmonics of type α. This constitutes the first step: *harmonic expansion* of fields (§ 2). Then, in § 3 we show that the space of harmonics of a given type α (which are still fields *on* E) can be interpreted as the space of sections of an appropriate 'effective' vector bundle $\tilde{\mathscr{F}}_\alpha$ *over* M. This is the second step: *dimensional reduction*. After that we consider the particularly interesting situation when $\mathscr{F}$ has a structure group, say, R; or, in other words, when $\mathscr{F}$ is a bundle associated to some principal bundle $U = U(E; \mathrm{R})$ via a certain representation ρ of R on F. The global symmetry group G is assumed to act on U by automorphisms. We use the results of [8, 9] to find out the effective ('dimensionally reduced') gauge group $\tilde{\mathrm{R}}$ and the effective principal bundle $\tilde{U}(M; \tilde{\mathrm{R}})$ over M. We also find the effective representation ρ_α of $\tilde{\mathrm{R}}$ on an appropriate F_α and show that the $\tilde{\mathscr{F}}_\alpha$ can also be constructed as the bundle associated to the effective principal bundle $\tilde{U}$ via ρ_α. In § 4 we apply these results to discuss the case of U being the bundle of orthonormal frames of E endowed with a G-invariant metric, and in § 5 we discuss tensor and spinor fields and dimensionally reduced Laplace and Dirac operators.

2. Harmonic expansion

Let E be a manifold with a given action of a compact Lie group G. For instance, E can be a multidimensional universe with G being its global symmetry group [6], but the analysis given here can be applied to many other situations as well. On E consider linear matter fields of some fixed type. If E is topologically trivial (e.g. if E is contractible (see [10, ch 4, cor 10.3])) then these matter fields can be described by vector-valued functions, i.e. by functions on E with values in some vector space F. In general, however, matter fields have to be described by cross sections of a certain (not necessarily trivial) vector bundle $\mathscr{F} = \mathscr{F}(E; F)$ with base E and typical fibre F (example: vector fields on S^2). We suppose F is a finite-dimensional vector space (real or complex). We also assume that the symmetry group G 'knows' not only how to act on E (by diffeomorphisms) but also, by bundle automorphisms, on $\mathscr{F}$†. Thus each $a \in \mathrm{G}$ maps a fibre F_y of $\mathscr{F}$ at $y \in E$ onto the fibre F_{ya} at $ya \in E$ by a one-to-one linear map: $F_y \ni u \to ua \in F_{ya}$ (we shall keep to the convention of using *right* actions of G on E and on $\mathscr{F}$).

Let $\Gamma\mathscr{F}$ denote the space of all cross sections ϕ of $\mathscr{F}$. Then $\Gamma\mathscr{F}$ is an infinite-dimensional vector space and its elements $\phi \in \Gamma\mathscr{F}$ are called fields, or field configurations. Having given the action of G on E and on $\mathscr{F}$ there is a natural representation of G on $\Gamma\mathscr{F}$. This induced representation T is defined by the formula

$$(T(a)\phi)(y) \doteq \phi(ya)a^{-1}. \tag{2.1}$$

† $\mathscr{F}$ is then called an equivariant vector bundle or, more specifically, a G-vector bundle (see, e.g., [11–13]).

We now want to decompose $\Gamma\mathscr{F}$—the space of field configurations—into invariant subspaces $\Gamma_\alpha\mathscr{F}$ corresponding to the irreducible representations α of the symmetry group G, with $\Gamma_\alpha\mathscr{F}$ consisting of field configurations transforming according to the irreducible representation α. But first let us introduce the notation which will be used throughout the rest of the paper. For each irreducible representation α let W_α† be the representation space of α (W_α is finite-dimensional since G is assumed to be compact), and let χ_α be the corresponding character. It is convenient to introduce $\hat{\mathrm{G}}$—the dual of G—which is the (discrete) space of (equivalence classes of) irreducible representations of G. Analogously as in [13, p 119] we can define, for every $\alpha\in\hat{\mathrm{G}}$, a projection operator $\pi_\alpha:\Gamma\mathscr{F}\to\Gamma\mathscr{F}$ by the formula

$$(\pi_\alpha\phi)(y) \doteq \mathrm{d}(\alpha)\int_{\mathrm{G}} (T(a)\phi)(y)\chi_\alpha(a^{-1})\,\mathrm{d}a \tag{2.2}$$

where $\mathrm{d}(\alpha)=\dim W_\alpha$, and $\mathrm{d}a$ is the normalised Haar measure on G. The range of π_α will be denoted by $\Gamma_\alpha\mathscr{F}$ and its elements will be called harmonics of type α. It can be shown, and the proof can be adapted from [11, 13], that the algebraic direct sum $\bigoplus_{\alpha\in\hat{\mathrm{G}}}\Gamma_\alpha\mathscr{F}$ is dense in $\Gamma\mathscr{F}$; more precisely, every field configuration $\phi\in\Gamma\mathscr{F}$ can be approximated with an arbitrary accuracy by a finite superposition of harmonics in the following sense: for every $\varepsilon>0$ and for every compact $K\subset E$ there is a finite subset $A\subset\hat{\mathrm{G}}$ and $\phi_A\in\bigoplus_{\alpha\in A}\Gamma_\alpha\mathscr{F}$ such that $\sup_{y\in K}\|\phi(y)-\phi_A(y)\|<\varepsilon$. The norm here is any continuous norm on (the fibres of) $\mathscr{F}$. This approximation theorem (which is akin to the classical Peter-Weyl theorem (see, e.g., [14])) is sufficient for our purposes; we do not need to investigate in what sense the 'inverse Fourier transform' formula

$$\phi(y)\sim\sum_{\alpha\in\hat{\mathrm{G}}}\pi_\alpha\phi(y)$$

holds. In particular we are not interested in making $\Gamma\mathscr{F}$ into a Hilbert space—this is because E is thought of as a model for (an extended) spacetime of a hyperbolic signature, and in field theory it is the space of *solutions* of a hyperbolic differential equation rather than the space of *all* field configurations that carries a useful Hilbert space structure.

It is a general philosophy of any application of the Fourier analysis that, when the group G is correctly identified, the physically most important are the lowest modes, i.e. harmonics corresponding to a few of the lowest dimensional representations of G. It is therefore reasonable, and the Peter-Weyl theorem makes such a restriction also mathematically founded, to concentrate on the analysis of the elements of Γ_α, with $\alpha\in\hat{\mathrm{G}}$ arbitrary but fixed‡. This is what we will do in the following.

Let w_i ($i=1,2,\ldots,d_\alpha$) be a basis in W_α and let w^i be the dual basis in W_α^*—the dual of W_α. We can now represent the χ_α factor in (2.2) as

$$\chi_\alpha(a^{-1})=\mathrm{Tr}(\alpha(a^{-1}))=\sum_i\langle w^i,\alpha(a^{-1})w_i\rangle=\sum_i\langle\alpha^*(a)w^i,w_i\rangle \tag{2.3}$$

where $\alpha^*(a)\doteq\alpha(a^{-1})^T$ is the contragradient representation of G on W_α^*. It is therefore natural to introduce, for each $\phi\in\Gamma\mathscr{F}$ and each $w\in W_\alpha^*$, *the Fourier component of* ϕ

† Unless stated otherwise, F and W_α are assumed to be *complex* vector spaces.

‡ It may happen that such a restriction is incompatible with field equations in E which may imply the necessity of having infinite towers of representations (see e.g. [15]). In such a case a reasonable approximation is needed to cut the tower (consistency problem).

along w by

$$\phi_w(y) = d(\alpha) \int [T(a)\phi](y)\alpha^*(a) w \, da. \tag{2.4}$$

It follows then from (2.2)-(2.4) that

$$\pi_\alpha \phi = \sum_i \langle \phi_{w^i}, w_i \rangle \tag{2.5}$$

w_i being any basis in W_α. In other words, ϕ can be restored by knowing its Fourier components ϕ_w. We will now express this fact more precisely, introducing at the same time the concepts we shall need later on.

We denote by $\mathscr{F}_\alpha$ the fibre bundle fibres of which are the tensor products $F_y \otimes W^*_\alpha$ of fibres F_y of $\mathscr{F}$ and W^*_α. $\mathscr{F}_\alpha$ can also be thought of as the tensor product $\mathscr{F} \otimes \boldsymbol{W}^*_\alpha$ of two vector bundles: $\mathscr{F}$ and the trivial bundle $\boldsymbol{W}^*_\alpha = E \times W^*_\alpha$. The Fourier components (or field harmonics) ϕ_w defined in (2.4) are now cross sections of the bundle $\mathscr{F}_\alpha$. We denote by $T \otimes \alpha^*$ the representation of G on $\Gamma\mathscr{F}_\alpha$† defined by

$$[(T \otimes \alpha^*)(a)\phi \otimes w](y) \doteq \phi(ya)a^{-1} \otimes \alpha^*(a)w(y). \tag{2.6}$$

This is a particular case of the general formula (2.1) provided that the action of G on the trivial bundle W^*_α is defined by

$$(y, w)a \doteq (ya, \alpha^*(a^{-1})w). \tag{2.7}$$

Comparing now the formulae (2.2) and (2.4), with T replaced by $T \otimes \alpha^*$ and $\chi_\alpha(a)$ put identically 1, we find that ϕ_w are G-invariant sections of $\mathscr{F}_\alpha$. We call $\Gamma_{\text{inv}}\mathscr{F}_\alpha$ the space of these sections. Explicitly, $\phi \in \Gamma_{\text{inv}}\mathscr{F}_\alpha$ if and only if $\phi: E \ni y \to \phi(y) \in F_y \otimes W^*_\alpha$ and $\phi(ya) = \phi(y)a$, where the right action of G on $\mathscr{F} \otimes W^*_\alpha$ is the product of the original action of G on $\mathscr{F}$ and of (2.7). One proves then (see [11, 13, 16]) that (2.5) gives an isomorphism between $\Gamma_\alpha\mathscr{F}$ and $W_\alpha \otimes \Gamma_{\text{inv}}\mathscr{F}_\alpha$‡. Because of this fact the term 'harmonics' will also be used for the elements of $\Gamma_{\text{inv}}\mathscr{F}_\alpha$.

We summarise the discussion: an arbitrary field configuration ϕ can be expanded into harmonics according to the irreducible representations of G; we have

$$\Gamma\mathscr{F} \sim \bigoplus_{\alpha \in \hat{G}} W_\alpha \otimes \Gamma_{\text{inv}}\mathscr{F}_\alpha \tag{2.8}$$

where $\mathscr{F}_\alpha = \mathscr{F} \otimes W^*_\alpha$ and $\Gamma_{\text{inv}}\mathscr{F}_\alpha$ is the space of G-invariant sections of the bundle $\mathscr{F}_\alpha$§.

3. Dimensional reduction of a G-vector bundle

We have seen that the natural building blocks out of which every field configuration can be reconstructed are harmonics, i.e. G-invariant (or 'equivariant') sections of vector bundles $\mathscr{F}_\alpha = \mathscr{F} \otimes W^*_\alpha$, $\alpha \in \hat{G}$. The discussion which will now follow can be applied to the space of invariant sections of any equivariant vector bundle. Therefore we consider here $\Gamma_{\text{inv}}\mathscr{F}$, $\mathscr{F}$ being an arbitrary equivariant vector bundle over E, and at the end we shall apply the results replacing $\mathscr{F}$ by $\mathscr{F}_\alpha$.

† Observe that $\Gamma(\mathscr{F} \otimes \boldsymbol{W}^*_\alpha)$ is naturally isomorphic to $\Gamma\mathscr{F} \otimes \boldsymbol{W}^*_\alpha$—thus, one can omit brackets in $\Gamma\mathscr{F} \otimes \boldsymbol{W}^*_\alpha$; also, we will often write W^*_α instead of $\boldsymbol{W}^*_\alpha$.

‡ When $\mathscr{F}$ is a *real* vector space and α is a *real irreducible* representation, then $\Gamma_\alpha\mathscr{F}$ is isomorphic to $W_\alpha \otimes' \Gamma_{\text{inv}}\mathscr{F} \otimes W^*_\alpha$, where the tensor product $\otimes'$ is taken over the commuting ring of α which is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (see [11] and also [17] for the concept of a commuting ring).

§ Observe that $\Gamma_{\text{inv}}\mathscr{F}_\alpha \simeq \text{Hom}_G(\Gamma\mathscr{F}, W^*_\alpha)$. Thus $\Gamma_\alpha\mathscr{F} \simeq W_\alpha \otimes \text{Hom}_G(\Gamma\mathscr{F}, W^*_\alpha)$—the fact also known as a Frobenius reciprocity theorem.

Let us recall the situation we are dealing with: G—a compact Lie group—acts on a manifold E and on a vector bundle $\mathscr{F} = \mathscr{F}(E; F)$ over E. In the following we shall always assume that E is a *simple* G space, i.e. that the stability groups of the points of E are all conjugated to a standard one, denoted by H. Every orbit is then isomorphic to G/H and the space M of all orbits is a manifold. With this preparation the very essence of the dimensional reduction mechanism of matter fields can be expressed by the following result.

Theorem 3.1. There is a natural isomorphism between the space $\Gamma_{\mathrm{inv}}\mathscr{F}$ of *invariant* sections of an equivariant vector bundle $\mathscr{F}$ *over* E and the space of *all* sections of a certain vector bundle $\tilde{\mathscr{F}}$ *over* M.

A proof of this result can be found in [18]. Here we indicate only how the bundle $\tilde{\mathscr{F}}$ is built out. The construction goes in two steps: first an invariant subbundle $\mathscr{F}_0$ of $\mathscr{F}$ is constructed, and next $\tilde{\mathscr{F}}$ is defined as the quotient $\mathscr{F}_0/\mathrm{G}$. To define $\mathscr{F}_0$ denote by G_y the stability group at $y \in E$. Then G_y acts by endomorphisms on F_y, and $(F_y)_0$ is defined as the subspace of F_y consisting of G_y-invariant vectors. The collection $\mathscr{F}_0 = \bigcup_y (F_y)_0$ is a G-invariant subbundle of $\mathscr{F}$. One then shows that the quotient $\tilde{\mathscr{F}} \doteq \mathscr{F}_0/\mathrm{G}$ can be given the structure of a vector bundle over $M = E/\mathrm{G}$.

As was said at the beginning of this section this construction is now to be applied to the bundles $\mathscr{F}_\alpha = \mathscr{F} \otimes W^*_\alpha$ rather than to $\mathscr{F}$ itself, although the $\mathscr{F}$ of § 2 is a particular case of the $\mathscr{F}_\alpha$ family, namely $\mathscr{F} = \mathscr{F}_\alpha$ for α—the trivial representation of G.

Although the above recipe is rather straightforward, it has one serious disadvantage: it does not tell us how the effective structure group of $\tilde{\mathscr{F}}_\alpha$ is to be found when the initial structure group of $\mathscr{F}$ is known. The rest of this section will be devoted to an alternative construction of the $\tilde{\mathscr{F}}_\alpha$ which will make the problem of reduction of the structure (or 'gauge') group explicit. In the construction below, as well as in the rest of this paper, we shall exploit the results of our earlier work [7–9]. For the reader's convenience let us first briefly recall the concepts and results which will be needed in the following.

We start, as before, with a simple G space E, G being a compact Lie group. Let P be the submanifold of E consisting of all points with the stability group H (assumed connected):

$$P = \{y \in E : \mathrm{G}_y = \mathrm{H}\}.$$

One proves then that P is a principal bundle over M with structure group $\mathrm{N(H)|H}$, N(H) being the normaliser of H in G: $\mathrm{N(H)} = \{a \in \mathrm{G}\colon a\mathrm{H}a^{-1} = \mathrm{H}\}$. Suppose now $U = U(E; \mathrm{R})$ is a principal bundle over E with structure group R, and let G act also on U by principal bundle automorphisms. (Such a situation is, for instance, studied when considering symmetries of Yang–Mills fields.) The action of G on U is characterised by a certain homomorphism $\lambda : \mathrm{H} \to \mathrm{R}$ [8]. By applying the same method which was used to define P, but now replacing E by U and G by $\mathrm{R} \times \mathrm{G}$ one constructs a submanifold $\tilde{U} \subset U$ which is a principal bundle over M with the effective structure group $\tilde{\mathrm{R}} = \mathrm{N(H)|H}$, where $\mathrm{H} \doteq \mathrm{diag}(\lambda(\mathrm{H}), \mathrm{H}) \subset \mathrm{R} \times \mathrm{G}$ is the standard stability group for the $\mathrm{R} \times \mathrm{G}$ action on U. One also proves [8] that $\tilde{\mathrm{R}}$ is locally a product of $\mathrm{N(H)|H}$ and the centraliser Z of $\lambda(\mathrm{H})$ in R†.

† The earlier papers dealing with the problem of symmetric gauge fields [19–21] did not consider the N(H)|H term.

With the above information at hand let us consider again our vector bundle $\mathscr{F}$, but now assuming it to have a structure group R. Or, in other words, we assume there is given a representation ρ of R on a vector space F, and $\mathscr{F}=\mathscr{F}(E;F)$ is the bundle associated to U via the representation ρ. The standard notation is $\mathscr{F}=U\times_R F$ or, better $U\times_\rho F$. We remind the reader (for more information see [22, 23]) that $\mathscr{F}$ is constructed out of the cartesian product $U\times F$ by identifying those pairs (u, v) and (u', v') for which $u'=ur$ and $v'=\rho(r^{-1})v$ for some $r\in \mathrm{R}$. The equivalence class of (u, v) is conveniently denoted as $u\cdot v$, and we have $ur\cdot v=u\cdot\rho(r)v$, which gives a formal justification for the notation used. We interpret $u\cdot v$ as 'the object having coordinates v in the frame u'. Observe that the action of G on U induces, in a natural way, the action of G on $\mathscr{F}$:

$$(u\cdot v)a\doteq ua\cdot v.$$

Thus $\mathscr{F}$ is automatically an equivariant vector bundle and the construction given by theorem 3.1 can be applied to $\mathscr{F}_\alpha$ to give the vector bundle $\tilde{\mathscr{F}}_\alpha$ over M. But now we are prepared to describe an alternative construction of $\tilde{\mathscr{F}}_\alpha$: we shall construct it as a bundle associated to $\tilde{U}$ via a certain representation ρ_α of $\tilde{\mathrm{R}}$ on a vector space F_α which we presently describe. First of all observe that $\rho\otimes\alpha^*$ is a representation of R×G on $F\otimes W^*_\alpha$. Denote by $F_\alpha=F\otimes_{\mathrm{H}} W^*_\alpha$ the subspace of $F\otimes W^*_\alpha$ consisting of vectors invariant under the subgroup $\mathbf{H}=\mathrm{diag}(\lambda(\mathrm{H}), \mathbf{H})\subset \mathrm{R}\times\mathrm{G}$. Then $\mathscr{F}_\alpha$ is invariant (as a set) under the action of $\mathrm{N}(\mathbf{H})$ and, since $\mathbf{H}$ acts on $F\otimes_{\mathrm{H}} W^*_\alpha$ trivially, we effectively obtain a representation of $\mathrm{N}(\mathbf{H})|\mathbf{H}$ on F_α. This representation will be denoted by ρ_α. We then have the following.

Theorem 3.2. The bundle $\tilde{\mathscr{F}}_\alpha$ constructed out of $\mathscr{F}_\alpha$ according to theorem 3.1 is naturally isomorphic to the bundle $\tilde{U}\times_{\rho_\alpha} F_\alpha$ associated to $\tilde{U}$ via the representation ρ_α of $\tilde{\mathrm{R}}$ on F_α.

The only interesting part of a proof of this theorem is to show how the isomorphism in question is constructed: given $q\cdot v\in\tilde{U}\times_{\rho_\alpha} F_\alpha$ we have $q\in U$ and $v\in F_\alpha\subset F\otimes W^*_\alpha$, so that we can consider $q\cdot v$ as an element of $\mathscr{F}\otimes W^*_\alpha$. It is easy to check that $q\cdot v$ is in fact in $(\mathscr{F}\otimes W^*_\alpha)_0$ (see the discussion following theorem 3.1). The isomorphism is then given by the map $\tilde{U}\times_{\rho_\alpha} F_\alpha\ni q\cdot v\to(q\cdot v)\mathrm{G}\in(\mathscr{F}\otimes W^*_\alpha)_0/\mathrm{G}$, where $(q\cdot v)\mathrm{G}$ denotes the orbit (i.e. equivalence class under the quotient map) of $q\cdot v$ under the action of G on $(\mathscr{F}\otimes W^*_\alpha)_0$. It is easy to see that the map above is well defined, i.e. the result does not depend on the choice of representatives q and v in the class $q\cdot v\in\tilde{U}\times_{\rho_\alpha} F_\alpha$.

We end this section with a brief discussion of dimensional reduction of linear differential operators acting on the fields. Here we have the following result.

Theorem 3.3. Every G-invariant linear differential operator D on $\Gamma\mathscr{F}$ induces a linear differential operator $\tilde{D}_\alpha$ on $\tilde{\mathscr{F}}_\alpha$.

The construction of $\tilde{D}_\alpha$ is straightforward: one first extends D to $D_\alpha=D\otimes I$ on $\Gamma\mathscr{F}\otimes W^*_\alpha$ and observes that $\Gamma_{\mathrm{inv}}(\mathscr{F}\otimes W^*_\alpha)$ is invariant under $D\otimes I$. But $\Gamma_{\mathrm{inv}}(\mathscr{F}\otimes W^*_\alpha)$ is isomorphic to $\Gamma\tilde{\mathscr{F}}_\alpha$ by theorem 3.1. Thus $\tilde{D}_\alpha$ is defined as the image of D_α, restricted to $\Gamma_{\mathrm{inv}}(\mathscr{F}\otimes W^*_\alpha)$, under this isomorphism. It remains to prove that $\tilde{D}_\alpha$ so defined is again a *differential* operator, and a proof of this fact can be found in [18].

4. Dimensional reduction of the bundle of orthonormal frames

So far we considered a quite general abstract vector bundle $\mathscr{F}$ over E. In particular we had to assume that some action of G on $\mathscr{F}$ is somehow given—it had to agree with the action of G on E but it did not follow from the latter. In the following we shall focus our discussion on the particular case of $\mathscr{F}$ being the bundle of tensors or spinors on E. For this we have to apply the reduction process $U \to \tilde{U}$ in a particular case of U being the bundle of orthonormal (or spin) frames. It is important to notice that the action of G on E essentially determines its action on tensors and spinors. But first we have to discuss in some details Riemannian geometry of simple G spaces. We recall here the relevant concepts and facts; more information can be found in [6–8].

Let us assume that E is equipped with a G-invariant metric g_E (we shall assume that the induced metric on orbits of G in E is either positive or negative definite). Then, because of G invariance, g_E generates a metric g_M on $M = E/\mathrm{G}$, gauge field A_μ on M with gauge group N(H)|H, and a certain number of scalar fields on M. The Lie algebra $\mathscr{G}$ can be decomposed into $\mathscr{G} = \mathscr{H} + \mathscr{K} + \mathscr{L}$, where $\mathscr{N} \doteq \mathscr{H} + \mathscr{K}$ is the Lie algebra of N(H), $\mathscr{S} \doteq \mathscr{K} + \mathscr{L}$ can be identified with the tangent space to $S = \mathrm{G}/\mathrm{H}$ at the origin, and both decompositions $\mathscr{G} = \mathscr{N} + \mathscr{L}$ and $\mathscr{G} = \mathscr{H} + \mathscr{S}$ are reductive. We recall that H is assumed to be connected. Then $\mathscr{K}$, which can be identified with the Lie algebra of $K \doteq \mathrm{N(H)|H}$, can also be characterised as the space of Ad(H)-invariant elements of $\mathscr{S}$.

We denote by $\mathrm{O}E$ the bundle of orthonormal frames of E. The structure group of $\mathrm{O}E$ will be denoted by $\mathrm{O}(\eta_E)$ or, shortly, $\mathrm{O}(E)$. It consists of matrices Λ satisfying $\Lambda^T \eta_E \Lambda = \eta_E$ where η_E is the canonical diagonal form of g_E; if E is a multidimensional universe then it is natural to take $\eta_E = \pm\mathrm{diag}(-1, +1, +1, \ldots, +1)$. Since G acts on E by isometries the action of G on E lifts automatically to $\mathrm{O}E$; indeed, point transformations induce transformations of tangent vectors and of frames and, being isometric, they map orthonormal frames into orthonormal ones. But $\mathrm{O}E$ and $\mathrm{O}(E)$ are not yet our U and R of § 3—the bundle $\mathrm{O}E$ is unnecessarily large and we shall first use the extra information we have to reduce its structure group. This can be done as follows: for every $p \in P$ define $\mathscr{K}_p$ and $\mathscr{L}_p$ as the subspaces of T_pE spanned by the Killing vectors from $\mathscr{K}$ and $\mathscr{L}$, respectively. For an arbitrary $y \in E$ there always exist $p \in P$ and $a \in \mathrm{G}$ such that $y = pa$; we define $\mathscr{K}_y \doteq \mathscr{K}_p a$ and $\mathscr{L}_y \doteq \mathscr{L}_p a$. Because Ad(H) leaves the subspaces $\mathscr{K}$ and $\mathscr{L}$ of $\mathscr{G}$ invariant (with $\mathscr{K}$ being even pointwise invariant), our definitions are unambiguous.

Definition 4.1. An orthonormal frame $e_A(y)$ at $y \in E$ is called *adapted* if $e_A = (e_\mu, e_{\hat{a}}, e_a)$ with $e_{\hat{a}} \in \mathscr{K}_y$, $e_a \in \mathscr{L}_y$ and e_μ orthogonal to $\mathscr{S}_y \doteq \mathscr{K}_y \oplus \mathscr{L}_y$†.

The set of all adapted orthonormal frames will be denoted by $\mathrm{AO}E$—it is a principal bundle with structure group $\mathrm{AO}(E) = \mathrm{O}(M) \times \mathrm{O}(K) \times \mathrm{O}(L)$, where $\mathrm{O}(M)$ is the (pseudo-)orthogonal group of η_M—the standard flat metric of M, while $\mathrm{O}(K)$ and $\mathrm{O}(L)$ are orthogonal groups in $k = \dim(\mathscr{K})$ and $l = \dim(\mathscr{L})$ dimensions, respectively. The group G acts on $\mathrm{AO}E$ by automorphisms. It is only now that we shall apply the dimensional reduction mechanism discussed in § 3 by putting $U \equiv \mathrm{AO}E$ and $R \equiv \mathrm{AO}(E)$; but first we have to identify the homomorphism λ: $\mathrm{H} \to \mathrm{R}$. Let T_i be a basis in $\mathscr{G}$ with $T_{\hat{a}} \in \mathscr{K}$, $T_a \in \mathscr{L}$ and $T_\alpha = (T_{\hat{a}}, T_a)$. We denote by $\varepsilon_i(y)$ the fundamental vector

† For $p \in P$ the vectors $e_\mu(p)$ span the horizontal subspace of the induced N(H)|H connection A_μ.

fields on E (Killing vectors) generated by T_i. The vectors $\varepsilon_\alpha(y)$ then span the vertical subspace $\mathscr{S}_y$, provided y is not too far from P. In a neighbourhood of P we can therefore always write

$$e_\alpha(y) = \varepsilon_\beta(y)\Phi^\beta{}_\alpha(e; y) \tag{4.1}$$

and if e_α is replaced by $e'_\alpha = e_\beta \Lambda^\beta{}_\alpha$ then $\Phi(e\Lambda) = \Phi(e)\Lambda$†. For $e_A \in \mathrm{AO}E$, $\Phi(e)$ has the block diagonal form $(\Phi^\alpha{}_\beta) = (\Phi^{\hat{a}}{}_{\hat{b}}, \Phi^a{}_b)$. Now, let $e(p) \in \mathrm{AO}E$. Then the action of any $h \in \mathrm{H}$ leaves p invariant and thus rotates $e_A(p)$ by an orthogonal transformation; in fact it is only the e_a which are rotated. Thus we have

$$e_a h^{-1} = e_b \lambda(e; h)^b{}_a \qquad h \in \mathrm{H}$$

$\lambda : \mathrm{H} \to \mathrm{O}(L)$ being a group homomorphism. Observe that

$$\lambda(e\Lambda; h) = \Lambda^{-1}\lambda(e; h)\Lambda.$$

Let $A^i{}_j$ be the matrix of the adjoint representation of G on $\mathscr{G}$. Then, since $\varepsilon h^{-1} = \varepsilon A(h)$, we find that

$$\lambda(e; h) = \Phi^{-1}(e)A(h)\Phi.$$

Although it is by no means necessary, it is very convenient to choose a basis T_i in $\mathscr{G}$ in such a way that, for some $p_0 \in P$, the Killing vectors $\varepsilon_\alpha(p_0)$ are orthonormal. Then $\lambda(\varepsilon(p_0); h) = A(h)$ is the matrix of the adjoint representation of H in $\mathscr{S}$. We now apply the results discussed in § 3 to $U = \mathrm{AO}E$, $R = \mathrm{AO}(E)$ and $\lambda = A$ to get the following result.

Theorem 4.1. The effective 'gauge' group $\mathrm{A\tilde{O}}(E)$ is locally isomorphic to the product $\mathrm{N(H)|H} \times \mathrm{O}(M) \times \mathrm{O}(K) \times Z$, where Z is the centraliser of Ad(H) in $\mathrm{O}(L)$.

Although we will not need this fact, it is worthwhile to notice that one can easily get rid of the $\mathrm{O}(K)$ factor of the effective gauge group. Indeed, we can always reduce the bundle by demanding that $e_{\hat{a}}(p)$ are fixed, e.g. by a standard orthonormalisation process of $\varepsilon_{\hat{a}}(p)$. That means that the effective internal gauge group is $\mathrm{N(H)|H} \times_{\mathrm{loc}} Z$. In the particular, well known, case when E is a principal bundle, the internal spaces are group manifolds, H is trivial, $\mathscr{S}$ is trivial, Z is trivial and $\mathrm{N(H)|H} = \mathrm{G}$ as expected.

According to the algorithm of the dimensional reduction discussed in § 3 (for more details see [7–9]) the reduced bundle $\mathrm{A\tilde{O}}E$ consists of those pairs $(p, e_A(p))$ for which $\lambda(e; h) = A(h)$. The following result gives an explicit description of the reduced frame bundle.

Theorem 4.2. The frames $e_A(p) \in \mathrm{A\tilde{O}}E$ are characterised by Φ satisfying

(i) $\Phi A(h) = A(h)\Phi$,

(ii) $\Phi^T g \Phi = \eta$

where $g(p) = (g_{\alpha\beta})$ are the components of the metric in the ε basis, and $\eta = \pm I$ depending on whether the metric in the orbit is positive or negative definite. Such frames exist for every $p \in P$ and any two such frames at the same $p \in P$ are connected by an orthogonal transformation $\Lambda_1 \times \Lambda_2$ with $\Lambda_1 \in \mathrm{O}(K)$ and $\Lambda_2 \in Z$, Z being the centraliser of Ad(H) in $\mathrm{O}(L)$.

† The indices $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, $\underset{\sim}{a}$, $\underset{\sim}{b}$, etc., will correspond to an orthonormal internal frame $e_{\underset{\sim}{\alpha}}$. Thus $\phi^\beta{}_{\underset{\sim}{\alpha}}$ is an internal vielbein and $\phi^{\underset{\sim}{\beta}}{}_\alpha$ is its inverse. We have $g_{\alpha\beta}\phi^\alpha{}_{\underset{\sim}{\gamma}}\phi^\beta{}_{\underset{\sim}{\delta}} = \eta_{\underset{\sim}{\gamma}\underset{\sim}{\delta}}$, $\eta^{\underset{\sim}{\gamma}\underset{\sim}{\delta}}\phi^\alpha{}_{\underset{\sim}{\gamma}}\phi^\beta{}_{\underset{\sim}{\delta}} = g^{\alpha\beta}$, etc.

Proof. The non-evident part of the statement is the existence result. To prove the existence of a Φ satisfying (i) and (ii) one starts with an arbitrary adapted frame (i.e. the result of a standard orthonormalisation of ε), with $e = \varepsilon\Phi$ (we omit the non-interesting e_μ vectors and observe that Φ satisfies (ii)) and $\lambda(e) = \Phi^{-1}A\Phi$ is an orthogonal matrix. It follows that $\Phi\Phi^T$ commutes with $A(h)$. Let $\Phi = \Phi_0\Lambda$ be the polar decomposition of Φ, with $\Phi_0 > 0$ and Λ orthogonal. Then $\Phi_0 = (\Phi\Phi^T)^{1/2}$ commutes with $A(h)$, and it also satisfies (ii) since $\Phi_0 = \Phi\Lambda^{-1}$ and Λ is an orthogonal transformation.

With the information given by the above theorem we can also explicitly describe the reduced structure group $A\tilde{O}E$. Indeed, let $e(p)$ and $e(p')$ be in $A\tilde{O}E$, p' and p being in the same fibre of P. Then we first choose $n \in \mathrm{N(H)}$ such that $p' = pn$, and then define $\Lambda \in \mathrm{O}(L)$ by the relation

$$e(p)n = e(p')\Lambda^{-1}$$

(we again omit e_μ and $e_{\hat{a}}$, which are uninteresting). Since both e and e' are in $A\tilde{O}E$, Λ must satisfy the constraint that $A(n^{-1})\Lambda$ commutes with $A(\mathrm{H})$ on $\mathscr{L}$. But n is unique only modulo $h \in \mathrm{H}$. Thus (n, Λ) should be identified with $(hn, A(h)\Lambda)$. Consequently we have the following result.

Theorem 4.3. The effective structure group $A\tilde{O}(E)$ is the product $\mathrm{O}(M) \times \mathrm{O}(K) \times \mathrm{R}_1$, where R_1 consists of pairs (n, Λ) with $n \in \mathrm{N(H)}$ and $\Lambda \in \mathrm{O}(L)$ such that $A(n^{-1})\Lambda$ commutes with $A(\mathrm{H})$; the pairs (n, Λ) and $(hn, A(h)\Lambda)$, $h \in \mathrm{H}$, define the same element of R_1.

We can now interpret the Φ as a cross section of a bundle associated to $A\tilde{O}E$. Indeed, with $e \in A\tilde{O}E$ and (n, Λ) as in theorem 4.3 we find

$$\Phi(en\Lambda) = A(n)^{-1}\Phi(e)\Lambda$$

and the condition (i) of theorem 4.2 ensures that the above transformation law effectively depends on the equivalence class of (n, Λ) only.

5. Dimensional reduction of tensor and spinor fields

For calculating the dimensionally reduced Laplace and Dirac operators we shall need explicit expressions for the affine connection $\omega_A{}^B{}_C(p)$, $p \in P$. These expressions will, in general, depend on a moving frame e_A which we will choose in $A\tilde{O}E$ at the points of P. More specifically we proceed as follows.

(i) Choose a local cross section $\sigma: x \to \sigma(x)$ of the bundle P.

(ii) At every point $\sigma(x) \in P$ choose an adapted frame $e_A(\sigma(x)) \in A\tilde{O}E$. Then $(\sigma(x), e_A(\sigma(x)))$ is a local cross section of the bundle $A\tilde{O}E$.

(iii) Extend $e_A(\sigma(x))$ to an open neighbourhood of σ in E defining $e_A(\sigma(x)e^\xi) \doteq e_A(\sigma(x))e^\xi$ for ξ running through a neighbourhood of zero in $\mathscr{G}$.

With the moving frame e_A introduced as above we write $e_\alpha = \varepsilon_\beta\Phi^\beta{}_\alpha$ as in (4.1), and from (iii) we easily find

$$\varepsilon_\alpha(\Phi^\beta{}_\delta) = -C^\beta_{\alpha\gamma}\Phi^\gamma{}_\delta. \tag{5.1}$$

Observe that now e_μ is a horizontal (i.e. orthogonal to the orbits of G in E) lift of a certain moving frame, denoted by ∂_μ, on M. The Christoffel symbols in the $(e_\mu, \varepsilon_\alpha)$ basis have been given in [7]. Here we have to transform them to an orthonormal basis (e_μ, e_α). The results of a straightforward calculation read as follows ($\omega_{\mu,\nu\sigma}$ the affine connection of M in the ∂_μ basis†),

$$\begin{aligned}
&\omega_{\mu,\alpha\nu} = -\omega_{\mu,\nu\alpha} = \omega_{\alpha,\mu\nu} = -\tfrac{1}{2}(\eta\Phi^{-1})_{\alpha\beta}F^\beta_{\mu\nu} = -\tfrac{1}{2}\eta_{\alpha\beta}\phi^\beta{}_\gamma F^\gamma_{\mu\nu}\\
&\omega_{\alpha,\beta\mu} = -\omega_{\alpha,\mu\beta} = \tfrac{1}{2}(\Phi^T(D_\mu g)\Phi)_{\alpha\beta} = \tfrac{1}{2}\phi^\gamma{}_\alpha D_\mu g_{\gamma\delta}\phi^\delta{}_\beta\\
&-\omega_{\mu,\alpha\beta} = \tfrac{1}{2}(\Phi^T(D_\mu g)\Phi)_{\alpha\beta} + (\eta\Phi^{-1}e_\mu(\Phi))_{\alpha\beta} = \tfrac{1}{2}\phi^\gamma{}_\alpha(D_\mu g_{\gamma\delta})\phi^\delta{}_\beta + \eta_{\alpha\gamma}\phi^\gamma{}_\delta e_\mu(\phi^\delta{}_\beta)
\end{aligned} \tag{5.2}$$

where $F^\alpha_{\mu\nu} \neq 0$ for $\alpha = \hat{a}$ only, and

$$F^{\hat{a}}_{\mu\nu} = \partial_\mu A^{\hat{a}}_\nu - \partial_\nu A^{\hat{a}}_\mu + \tfrac{1}{2}C^{\hat{a}}_{\hat{b}\hat{c}}A^{\hat{b}}_\mu A^{\hat{c}}_\nu$$

is the field strength of the N(H)|H gauge field,

$$D_\mu g_{\alpha\beta} = \partial_\mu(g_{\alpha\beta}) + C^\delta_{\alpha\hat{c}}A^{\hat{c}}_\mu g_{\delta\beta} + C^\delta_{\beta\hat{c}}A^{\hat{c}}_\mu g_{\alpha\delta}$$

is the covariant derivative of $g_{\alpha\beta}$, and $e_\mu(\Phi)$ is given by

$$e_\mu(\Phi) = \partial_\mu(\Phi) - A^{\hat{a}}_\mu\varepsilon_{\hat{a}}(\Phi).$$

We also used the flat metric η_{AB} of E to lower the index of $\omega_A{}^B{}_C$: $\omega_{A,BC} = \eta_{BD}\omega_A{}^D{}_C$.

By theorem 4.3 the effective structure group $\mathrm{A\tilde{O}}E$ is the product $\mathrm{O}(M)\times\mathrm{O}(K)\times\mathrm{R}_1$, where $\mathrm{R}_1 \overset{\text{loc}}{\approx} \mathrm{N(H)|H}\times Z$. We have already identified the O(M) and N(H)|H parts of the effective gauge field—they are $\omega_\mu{}^\nu{}_\sigma$ and $A^{\hat{a}}_\mu$ respectively. It still remains to find the $\mathrm{O}(K)\times Z$ part. A general construction of the effective, dimensionally reduced gauge field has been given in [8]. The construction given there in proposition 3.5 is formal but the idea behind it is quite simple: the effective horizontal lift of a path from M to $\mathrm{A\tilde{O}}E$ is obtained by a composition of a horizontal lift from M to P and a horizontal lift from P to $\mathrm{A\tilde{O}}E$. With that geometrical picture it is easy to find the remaining $\mathrm{O}(K)\times Z$ part of the effective connection: with respect to the local moving frame chosen above we have $B^\alpha_{\mu\beta} = \omega_\mu{}^\alpha{}_\beta$, where we have identified the Lie algebra $(\mathrm{O}(K)\times Z)$ with a subalgebra of $\mathscr{S} = \mathscr{H} + \mathscr{L}$. Explicitly, one has

$$B^\alpha_{\mu\beta} = -\phi^\alpha{}_\gamma e_\mu\phi^\gamma{}_\beta - \tfrac{1}{2}\phi^\alpha{}_\gamma g^{\gamma\delta}(D_\mu g_{\delta\lambda})\phi^\lambda{}_\beta. \tag{5.3}$$

With this preparatory knowledge at hand we now consider several examples. We start with the simplest one.

5.1. Laplace operator on scalar fields

A (real) scalar field on E is a section of the trivial bundle $\mathscr{F} = E\times\mathbb{R}$ which is associated to the trivial representation of O(E) on $\mathbb{R}$. According to the discussion in § 3 the effective fibre bundle $\tilde{\mathscr{F}}_\alpha$ over M has the fibre $F_\alpha = F\otimes_{\mathrm{H}} W^*_\alpha = \mathbb{R}\otimes_{\mathrm{H}} W^*_\alpha = (W^*_\alpha)_0$, i.e. F_α consists of H singlets in W^*_α. The effective field on M therefore gets the index $\phi\to\phi_L$ from the representation space W^*_α and is constrained by

$$\alpha(h)^L{}_M\phi_L = \phi_M \qquad h\in\mathrm{H} \tag{5.4}$$

where $\alpha(h)^L{}_M$ is the matrix of the representation α. Knowing the connection coefficients $\omega_A{}^B{}_C$ one easily finds the effective Laplace operator acting on $\boldsymbol{\phi} = (\phi_L)$:

$$\Delta_{\text{eff}}\boldsymbol{\phi} = g^{\mu\nu}\mathscr{D}_\mu D_\nu\boldsymbol{\phi} + g^{\alpha\beta}T_\alpha T_\beta\boldsymbol{\phi} + v^\mu D_\mu\boldsymbol{\phi} \tag{5.5}$$

† Notice that in spite of the notation ∂_μ is an anholonomic orthonormal moving frame of M.

where $D_\mu \boldsymbol{\phi} = \partial_\mu(\boldsymbol{\phi}) + A_\mu^{\hat{a}} T_{\hat{a}} \boldsymbol{\phi}$ is the N(H)|H-covariant derivative, $\mathscr{D}_\mu$ contains also the gravitational part and v^μ is given by

$$v_\mu = \tfrac{1}{2}\,\mathrm{Tr}(g^{-1} D_\mu g) = \tfrac{1}{2}\,\mathrm{Tr}(g^{-1} \varepsilon_\mu(g)) \tag{5.6}$$

which measures a rate of change of the volume of the internal space. In the derivation of the above formulae we used the fact that G, being compact, is unimodular thus; in particular, $C^\alpha_{\alpha\beta} = 0$, C^i_{jk} being the structure constants of G. In the particular case when $g_{\alpha\beta}$ comes from a fixed Killing metric g_{ij} on G the effective mass term in (5.5) is nothing but the Casimir operator of G acting on $(W^*_\alpha)_0$. Indeed, owing to the constraint (5.4) we can extend the summation of the indices α, β in $g^{\alpha\beta} T_\alpha T_\beta$ to $g^{ij} T_i T_j$ which is the Casimir operator. In general, however, $g_{\alpha\beta}$ is non-constant and the term $D_\mu g_{\alpha\beta}$ contributes to the effective Laplacian.

5.2. *1-forms*

Here the vector bundle $\mathscr{F}$ is the cotangent bundle T^*E with the typical fibre $F = R^{N^*} = \mathbb{R}^{m^*} \oplus \mathbb{R}^{k^*} \oplus \mathbb{R}^{l^*}$. The effective fibre bundle $\tilde{\mathscr{F}}_\alpha$ over M has the fibre $F_\alpha = \mathbb{R}^{N^*} \otimes_{\mathrm{H}} W^*_\alpha = [\mathbb{R}^{m^*} \otimes (W^*_\alpha)_0] \oplus [\mathbb{R}^{k^*} \otimes (W^*_\alpha)_0] \oplus [\mathbb{R}^{l^*} \otimes_{\mathrm{H}} W^*_\alpha]$. In other words a 1-form ϕ_A on E gives rise to a multiplet $(\phi_{\mu L}, \phi_{\hat{a}L}, \phi_{aL})$ where each term has got an extra index from the representation space W^*_α constrained by $\alpha(h)^L{}_M \phi_{\cdot L} = \phi_{\cdot M}$ for the first two members of the multiplet and

$$\alpha(h)^L{}_M \phi_{aL} = A(h)^b{}_a \phi_{bM}$$

for the last one. Each member of the multiplet will, in general, split into submultiplets according to the colour charge of the gauge group $K = \mathrm{N(H)|H}$.

5.3. *Spinor fields*

The group $\mathrm{O}(E)$ is not simply connected and we denote by $\mathrm{Pin}(E)$ its two-fold simply connected spin covering. A spin structure† on E consists of a principal bundle $\mathrm{Pin}E$ over E with structure group $\mathrm{Pin}(E)$ and of a covering bundle homomorphism $\mathrm{Pin}E \to \mathrm{O}E$ commuting with the group homomorphism $\sigma: \mathrm{Pin}(E) \to \mathrm{O}(E)$. The elements of $\mathrm{Pin}E$ are called spin frames. Thus over each orthonormal frame $e \in \mathrm{O}E$ there sit two spin frames which differ by the transformation '-1' $\in \mathrm{Pin}(E)$. When a spin frame s is rotated by a transformation $\Lambda \in \mathrm{Pin}(E)$ then the corresponding orthonormal frame e is rotated by an orthogonal transformation $\sigma(\Lambda) \in \mathrm{O}(E)$. The group homomorphism σ is given explicitly by

$$\Lambda \Gamma_A \Lambda^{-1} = \Gamma_B \sigma(\Lambda)^B{}_A$$

where Γ_A are generators of the Clifford algebra of η_E satisfying $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$. As we have seen in § 4 the G-space structure of E allows us to reduce the bundle $\mathrm{O}E$ to the subbundle $\mathrm{AO}E$ of adapted frames, the structure group being reduced from $\mathrm{O}(E)$ to $\mathrm{AO}(E)$. Here we will take $\mathrm{AO}(E) = \mathrm{O}(M) \times \mathrm{O}(S)$ as there is no gain in a subtler reduction $\mathrm{O}(S) \to \mathrm{O}(K) \times \mathrm{O}(L)$. We denote by $\mathrm{APin}E$ the counterimage of $\mathrm{AO}E$ under the covering homomorphism. The elements of $\mathrm{APin}E$ are called adapted spin frames. Thus over each adapted orthonormal frame there sit two adapted spin frames. The

† For a discussion of spin structures see [24–26]. To avoid complications with orientation and time orientation we have replaced $\mathrm{Spin}_+(E)$ by $\mathrm{Pin}(E)$.

structure group of APin$E$ is APin(E) = (Pin(M) × Pin(S)/Z_2, where the quotient identifies the two 'minus identities' of Pin(M) and Pin(S) respectively†. Let $\partial_\mu(x)$ be a fixed orthonormal frame at $x \in M$ and consider the set of all spin frames at the points of the internal space S_x over x which project onto (e_μ, e_α) with e_μ being the horizontal lift of ∂_μ. This set of spin frames is a spin structure for S_x. Thus existence of a spin structure on E implies existence of a spin structure on the internal spaces S_x, $x \in M$, but not, in general, on M.

Let us assume that the action of G on AOE lifts to an action on APin$E$‡; then, since H is assumed to be connected, the homomorphism $\lambda: \mathrm{H} \to \mathrm{O}(S)$ induces a unique homomorphism $\tilde{\lambda}: \mathrm{H} \to \mathrm{Pin}(S)$. Repeating now the arguments used in § 4 we deduce that the resulting effective gauge group is now $(\mathrm{Pin}(M) \times \mathrm{N}(\mathbf{H})|\mathbf{H})/Z_2$ where $\mathbf{H} = \mathrm{diag}(\tilde{\lambda}(\mathrm{H}), \mathrm{H}) \subset \mathrm{Pin}(S) \times \mathrm{G}$. Therefore M is endowed with a generalisation of a Spin^c structure (for a discussion of Spin^c structures see, e.g., [29, 30]). To derive the dimensionally reduced Dirac operator we use a local moving spin frame corresponding to the orthonormal moving frame introduced at the beginning of this section. Let F be a representation space (real or complex) of the Clifford algebra $C(E)$. The generators of the representation of Pin(E) on F are then

$$\Sigma^{AB} = \tfrac{1}{4}[\Gamma^A, \Gamma^B]$$

and the Dirac operator is§

$$D_E = \Gamma^A(e_A + \tfrac{1}{2}\omega_{A,BC}\Sigma^{BC}).$$

The typical fibre F_α of the dimensionally reduced spinor of type α is then $F \times_{\mathrm{H}} W^*_\alpha$. Thus the spinor Ψ gets an extra index $\Psi = (\Psi_L)$ of the representation space W^*_α and satisfies the constraint

$$(\alpha^*(T_{\hat{\alpha}}) - \tfrac{1}{2}C_{\hat{\alpha},\beta\gamma}\Sigma^{\beta\gamma})\Psi = 0$$

where $T_{\hat{\alpha}}$ ($\hat{\alpha} = 1, 2, \ldots, \dim \mathrm{H}$) are the generators of $\mathscr{H}$. The dimensionally reduced Dirac operator can now be written down as

$$D^{\mathrm{eff}} = D_M + D_S + \Delta_1 + \Delta_2$$

with the four terms described as follows.

(i) D_M is the covariant Dirac operator on M. It is given by

$$D_M = \Gamma^\mu D_\mu$$

where D_μ is the covariant gravitational plus gauge derivative:

$$D_\mu = \partial_\mu + \tfrac{1}{2}\omega_{\mu,\nu\sigma}\Sigma^{\nu\sigma} + A^{\hat{a}}_\mu \alpha^*(T_{\hat{a}}) + \tfrac{1}{2}B_{\mu,\alpha\beta}\Sigma^{\alpha\beta}$$

with B_μ defined in (5.3).

(ii) D_s is the internal Dirac operator of type α:

$$D_s = \Gamma^\alpha(-\Phi^\beta{}_\alpha \alpha^*(T_\beta) + \tfrac{1}{2}\omega_{\alpha,\beta\gamma}\Sigma^{\beta\gamma}).$$

(iii) Δ_1 is the anomalous non-minimal interaction term‖

$$\Delta_1 = \tfrac{1}{4}\Gamma^\alpha \phi^\beta{}_\alpha F^\alpha_{\mu\nu}\Sigma^{\mu\nu}\eta_{\alpha\beta}.$$

† The action of the 'minus identity' on a spin frame corresponds to a 2π rotation of an orthonormal frame.
‡ Action of Lie groups on spin manifolds in a connection with a G index is discussed in [27]. See also [28] and [29] for U(1) actions.
§ For more information on the Dirac operator on manifolds, in particular, in a connection with the index theorem see [31–35]. Also [36] for the Dirac operator on G|H.
‖ For a discussion in case of Einstein–Cartan theory, principal bundle and invariant spinors, see e.g. [37].

(iv) Δ_2 is analogous to the last term of (5.5):

$$\Delta_2 = \tfrac{1}{2}\Gamma^\mu v_\mu$$

with v_μ given by (5.6).

Let us consider now the case of M four-dimensional of signature $(-+++)$ and S the typical internal space of signature $(++\ldots+)$. The Clifford algebra $C(E)$ of E is then isomorphic to a tensor product $C(E) = C(M)\otimes C(-S)$, the isomorphism being given by $\Gamma_\mu = \gamma_\mu \otimes I$, $\Gamma_\alpha = \gamma_5 \otimes \gamma_\alpha$ with $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ and $\{\gamma_\alpha, \gamma_\beta\} = -2\eta_{\alpha\beta}$, where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. Since $C(M) \simeq L(\mathbb{R}^4)$—the 4×4 real matrix algebra—it follows that the real spinor space F can be realised as the tensor product $F = \mathbb{R}^4 \otimes_{\mathbb{R}} F'$ where F' is a representation space of $C(-S)$. The simplest example is the five-dimensional Kaluza-Klein theory. Here S is one-dimensional and the Clifford algebra $C(-1)$ is isomorphic to the algebra $\mathbb{C}$ of complex numbers. Thus $F = \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^4$. The group G is, in this example, U(1) and all its irreducible representations α are realised on $\mathbb{C}$. Since H is now trivial the effective fibre is therefore $F_\alpha = F \otimes \mathbb{C} = \mathbb{C}^4$. The effective structure group is then $\mathrm{Pin}^c(3,1) \equiv (\mathrm{Pin}(3,1)\times \mathrm{U}(1))/Z_2$.

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