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Electromagnetic Permeability of the Vacuum and Light-Cone Structure*)

by

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Summary. It is shown that to give a constitutive tensor for the vacuum is to give a conformal structure to space-time.

To formulate laws of electromagnetism one first has to specify a constitutive tensor $\chi^{\alpha\beta}_{\mu\nu}$. Once χ is specified, Maxwell's equation reads: $dF=0$, and $d{*}F=j$, where

$$({*}F)_{\mu\nu}=\chi^{\alpha\beta}_{\mu\nu}F_{\alpha\gamma}.$$

We shall assume that, in a vacuum, the $*$-operator satisfies

(i) $$*w\wedge u=w\wedge *u, \qquad (\mathrm{u},\ w \text{ — two-forms})$$

(ii) $$*^2=-I.$$

Under these assumptions we shall prove that to give such a "$*$" is the same as to give a light-cone structure.

DEFINITION. Let E be a 4-dimensional real vector space, and let $(w_i)_{i=1,2,3}$ be a system of linearly independent bivectors $w_i\in \Lambda^2(E)$, such type $w_i\wedge w_j=0$ $(i,j=1,2,3)$. The system (w_i) is said to be of type I if and only if there exists $0\neq e\in E$, such that $w\wedge w_i=0$. Otherwise (w_i) is said to be of type II.

LEMMA 1. *A system (w_i) is of type I (resp. type II) if and only if there exists a basis $(e_\mu)_{\mu=0,1,2,3}$ in E such that*

(i) $$w_i=w_0\wedge e_i$$

resp., (ii) $w_i=\frac{\epsilon}{2}\,\varepsilon_{ijk}\,e_j\wedge e_k$, *where* $\epsilon=+1$ *or* -1..

Proof. Clearly (i) implies that (w_i) is of type I. Conversely, if (w_i) is of type I, and $e_0\wedge w_i=0$, then $w_i=w_0\wedge e_i$, and $(e_\mu)_{\mu=0,\dots,3}$ is a basis in E. Suppose now that

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(ii) is satisfied. If $e \wedge w_i = 0$, then $e = 0$, and so (w_i) is of type II. Conversely, let (w_i) be of type II. Since $w_1 \wedge w_2 = 0$, there exist linearly independent $f_i \in E$, such that $w_1 = f_2 \wedge f_3$, and $w_2 = f_3 \wedge f_1$. Let f_0 be any completion of (f_i) to a basis in E. Since $w_3 \wedge w_1 = 0$, so w_3 is of the form $w_3 = (af_2 + bf_3) \wedge \alpha^\mu f_\mu$. The coefficient a can not vanish, otherwise $f_3 \wedge w_i = 0$, contrary to the assumption. But $0 = w_2 \wedge w_3 = a\alpha^0 f_3 \wedge \wedge f_1 \wedge f_2 \wedge f_0$, and so $\alpha^0 = 0$. It follows that w_3 is of the form $w_3 = af_1 \wedge f_2 + bf_1 \wedge f_3 + + cf_2 \wedge f_3$, $a \neq 0$. Define

$$e_0 = f_0, \quad e_1 = |a|^{-1/2}(af_1 - cf_3)$$

$$e_2 = |a|^{-1/2}(af_2 + bf_3), \quad e_3 = |a|^{-1/2} f_3,$$

then

$$w_i = \frac{\epsilon}{2} \epsilon_{ijk}\, e_j \wedge e_k, \qquad \text{where } \epsilon = \operatorname{sgn}(a).$$

LEMMA 2. *Let $(w_i, \tilde{w}_j)_{i,j=1,2,3}$ be a basis in $A^2(E)$ such that*

(i) $$w_1 \wedge w_j = 0,$$

(ii) $$\tilde{w}_i \wedge \tilde{w}_j = 0,$$

(iii) $$w_i \wedge \tilde{w}_j = \delta_{ij} W, \quad \textit{for some} \quad 0 \neq W \varepsilon A^4(E).$$

Then exactly one of the systems (w_i), $(\tilde{w}_i)$ is of type II. If, say, (w_i) is of type II, then there exists a basis (e_μ) in E such that

(a) $$\varepsilon w_i = \frac{1}{2} \varepsilon_{ijk}\, e_j \wedge e_k,$$

(aa) $$\varepsilon \tilde{w}_i = e_0 \wedge e_i.$$

Proof. Suppose both (w_i), and $(\tilde{w}_i)$ are of type I. There are bases (e_μ), and (f_μ) such that $w_i = e_0 \wedge e_i$, and $\tilde{w}_i = f_0 \wedge f_i$. Let $f_\mu = a_\mu^\nu e_\nu$. Then $\tilde{w}_i = a_0^\mu a_i^\nu e_\mu \wedge e_\nu$, and so $\delta_{ij} W = w_i \wedge \tilde{w}_j = a_0^k a_l^j e_0 \wedge e_i \wedge e_k \wedge e_l$. By multiplying by a_0^i, we get $a_0^j = 0$, and so

$$\tilde{w}_i = a_0^0 a_i^\nu e_0 \wedge e_\nu = a_0^0 a_i^j e_0 \wedge e_j = a_0^0 a_i^j w_j,$$

contrary to the assumption of linear independence. We can therefore assume that (w_i) is of type II, and let (e_μ) be a basis for which (a) holds. Now, since $\tilde{w}_1 \wedge w_2 = 0$, $\tilde{w}_1$ is of the form $w_1 = (Ae_1 + Be_3) \wedge a^\mu e_\mu$. We have $A \neq 0$, otherwise $w_1 \wedge \tilde{w}_1 = 0$. We also have $a^0 \neq 0$, otherwise $\tilde{w}_1$ would be a linear combination of w_i. Now, $0 = \tilde{w}_1 \wedge w_3 = B\, a^0 e_3 \wedge e_0 \wedge e_1 \wedge e_2$, and so $B = 0$. It follows that $\tilde{w}_1$ is of the form $\tilde{w}_1 = a_1^\mu e_\mu \wedge e_1$. More generally, $w_i = a_i^\mu e_\mu \wedge e_i$. But now $\tilde{w}_i \wedge \tilde{w}_j = 0$ gives

$$a_i^\mu a_j^\nu e_\mu \wedge e_i \wedge e_\nu \wedge e_j = 0,$$

or

$$\varepsilon_{\mu i \nu j} a_i^\mu a_j^\nu = 0.$$

It follows that any three different indices i, j, k: $a_i^0 a_j^k = a_i^k a_j^0$, or

$$a_j^k / a_j^0 = a_i^k / a_i^0 = a^k.$$

We now have

$$a_i^0\,(a^k\,e_k+e_0)\wedge e_i=a_i^0\left(\frac{a_i^k}{a_i^0}+e_k+e_0\right)\wedge e_i=\tilde{w}_i.$$

Replacing e_0 by $e_0+a^k\,e_k$, we get $\tilde{w}_i=a_i^0\,e_0\wedge e_i$. Now, since $w_i\wedge\tilde{w}_i=w_j\wedge\tilde{w}_j$, we get $a_i^0=a_i^0\ {}^D\ a$. Replacing e_0 by $\varepsilon a e_0$ we finally get (aa).

LEMMA 3. *Let V be a $2N$-dimensional real vector space with a nondegenerate symmetric bilinear form (u,v). Let J be a symmetric linear operator in V with $J^2=-I$. There exist N vectors $w_1,\dots,w_N$ in V such that*

(i) $\quad (w_i, w_j)=0$

(ii) $\quad (e_i=Jw_j)=\delta_{ij}$ $\quad \Big\}\ i,j=1,\dots,n.$

The system $\{w_i, Jw_i\}$ is a basis for V.

Proof. Easy and standard.

THEOREM 1. *Let E be a 4-dimensional real vector space, and let $*: w\mapsto *w$ be a linear operator in $\Lambda^2(E)$ such that*

(i) $\quad *w\wedge u=w\wedge *u, \qquad u, w\in\Lambda^2(E)$

(ii) $\quad **=-I.$

There exists a basis (e_μ) in E such that

$$(*)\qquad *(e_\alpha\wedge e_\beta)=\frac{1}{2}\,\varepsilon_{\alpha\beta\sigma\rho}\,\eta^{\sigma\mu}\,\eta^{\rho\nu}\,e_\mu\wedge e_\nu,$$

where $\eta=\mathrm{diag}(1,-1,-1,-1)$.

Proof. Fix arbitrary $0\neq W\in\Lambda^4(E)$, and define a nondegenerate symmetric bilinear form (u,v) in $\Lambda^2(E)$ by $u\wedge v=(u.v)\,W$. Now, with $J=*$, the assumptions of Lemma 3 are satisfied, and so there exist bivectors w_i $(i=1,2,3)$ such that $w_i\wedge w_j=$ $=*w_i\wedge *w_j=0$, and $w_i\wedge *w_j=\delta_{ij}\,W$. With $\tilde{w}_i=*w_i$, the assumptions of Lemma 2 are satisfied. It follows that either (w_i) or $(*w_i)$ is of type II. If $(*w_i)$ is of type II, then there exists a basis (e_μ) such that

$$w_i=e_0\wedge e_i,\qquad *w_i=\frac{1}{2}\,\varepsilon_{ijk}\,e_j\wedge e_k,$$

and so, Theorem 1 holds. If, on the other hand, (w_i) is of type II, then the basis $(-e_0, e_i)$ satisfies the desired relation.

DEFINITION. A basis (e_μ) in E satisfying relation $(*)$ in Theorem 1 will be called a $*$-basis.

THEOREM 2. *Any two $*$ bases (e_μ) and $(\tilde{e}_\mu)$ are related by a transformation of the form*

$$\tilde{e}_\mu=\lambda e_\nu\,L^\nu_\mu,$$

where ${}^tL\eta L=\eta$, $\det(L)=+1$, *and* $\lambda>0$. *L and λ are uniquely determined by these conditions. Conversely, any such transformation transforms $*$-bases into $*$-bases.*

Proof. Assume both e_μ, and $\tilde{e}_\mu$ satisfy $(*)$, and let $\tilde{e}_\mu = e_\nu A^\nu_\mu$. This leads immediately to

$$\det(A)\, \varepsilon_{\kappa\lambda\mu\nu} = B^\alpha_\mu B^\beta_\lambda \varepsilon_{\mu\nu\alpha\beta},$$

where $B = \eta^{-1}\, {}^t A \eta A$. This implies $\det(A) > 0$, $B = \det(A)^{1/2} I$. It follows that ${}^t A \eta A = \det(A)^{1/2} \eta$, or with $\lambda = \det(A)^{-1/4}$, $A = \lambda L$, ${}^t L \eta L = \eta$, and $\det(L) = 1$. The rest of the theorem is obvious.

COROLLARY. *Given a $*$-operator as in Theorem* 1, *the set of all $*$-bases is a transitive homogeneous space for the proper conformal group CO* (1, 3).

It follows from the above corollary that a constitutive tensor of the vacuum equips space-time with a cone at each of its points. Or, equivalently, determines a Riemannian metric up to a scaling. This result has no analogue in more than four dimensions.

This note is a solution of a problem raised in a discussion with Professors R. Haag, D. Kastler and J. M. Singer.

А. Ядчик, **Электромагнитная проницаемость вакуума и структура световых конусов**

Содержание. В работе доказано, что определение комформной структуры пространства-времени эквивалентно определению тензора электромагнитной проинцаемости вакуума.