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Symmetry of Einstein - Yang - Mills systems and dimensional reduction

A. JADCZYK

Institute of Theoretical Physics, University of Wroclaw
50-205 Wroclaw, Cybulskiego 36, Poland

Abstract. *The following topics are discussed: G-invariant Riemannian metrics and principal connections, dimensional reduction of Einstein and Yang-Mills systems, curvature of coset spaces, dimensional reduction of spinors, geometrical interpretation of color and Higgs charges.*

1. INTRODUCTION

It is interesting to assume that space-time points are endowed with some internal structure. In modern language one assumes that our 4-dimensional space-time M is a base of a fiber bundle (E, π, M). E is a «multidimensional Universe» ($\dim E = 4 + N$), and $\pi : E \to M$ is a projection map identifying points in E which we do not discriminate. The idea that the events we normally perceive are only shadows, or projections, of things which take place in much more dimensions can be attributed to Plato. The fact that (under normal conditions) we are perfectly blind to the extra dimensions is naturally expressed by assuming that the fibers of E are homogeneous spaces. In this way certain symmetry group G is introduced, and it is tempting to connect this group with internal symmetry groups which are so helpful for classifying of elementary particles multiplets. An example of this type of a structure is given by a gauge theory when formulated in terms of fiber bundles. One starts there with a principal bundle $\pi : P \to M$, and the fibers of P are group manifolds. In electromagnetism ($G = U(1)$) the extra dimen-

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sion is related to an unobservable phase of a wave function. For non-Abelian gauge fields one still talks of a non-observable, non-integrable, phase factor [1], but it is no longer connected with representing of quantum mechanical states by rays (1-dimensional subspaces) rather than vectors. The attempts of introducing quaternionic Hilbert spaces, which are natural for $G = SU(2)$, produced so far no workable model, maybe because of the rigidity of our thinking. In this connection see [2], [3]. Although principal bundles proved to be an indispensable concept in discussing gauge fields and their interaction with matter, many people felt uneasy about an *ad hoc* introduction of such a very special geometrical structure. One way of a more general, and more natural, introducing of a fibration is by a dynamical mechanism called a «spontaneous compactification» (see e.g. [4], also [5] and references there). Exact geometrical meaning of this mechanism is not yet clear, and I shall focus here on a simpler idea which relates the fibration of E to a global action of some internal symmetry group G. A dynamical origin of G and its action on E is left open here. Keeping in mind the obvious shortcomings of our model it is nevertheless worthwhile to study it as a straightforward generalization of the principal bundle structure which proved to be already useful. Before we go into the details let us give first some relevant references. A unification of gravitation and electromagnetism ($U(1)$ gauge field) based on the idea of a five-dimensional Universe was worked out by Kaluza [6] and Klein [7]. A possibility of a non-Abelian generalization of this idea was discussed several times [8, 9, 10] and its full geometrical and dynamical content has been given in [11, 12, 13, 14]. In all these papers it was always assumed that E is a principal bundle, i.e. that the internal spaces are group manifolds. The only exception is the Souriau paper [8], where the sphere S^2 was proposed as a model for an internal space related to the isospin group $G = SU(2)$. A general framework of G-invariant dimensional reduction described below has been given by Coquereaux and Jadczyk [15].

For the convenience of the reader we include a selection of references (Ref. [35 - 48]) where a broader spectrum of problems and approaches to gauge fields and Kaluza-Klein theories is discussed.

2. MULTIDIMENSIONAL UNIVERSE AND ITS BUNDLE STRUCTURE

2.1. Assumptions and notation

As a mathematical model for a multidimensional Universe we take a manifold E (the Universe) on which a (global symmetry) group G acts as a group of transformations: We assume that

i) G is a compact Lie group

ii) G acts effectively on E from the right

iii) there is only one stratum (one orbit type).

We let $G_y = \{a \in G : ya = y\}$ to denote the isotropy group at y, and $G(y) = \{ya : a \in G\}$ to denote the orbit of G through y. Let $M = E/G$ be the space of orbits. One can consider M as a quotient of E by the following equivalence relation: $y \sim y'$ in E if and only if y and y' are connected by some symmetry transformation from G i.e. $y' = ya$, $a \in G$. The points of M are equivalence classes of this relation i.e. the orbits of G. There is a canonical projection $\pi : E \to M$ which sends every $y \in E$ into the equivalence class to which it belongs, i.e. into its orbit. By iii) all orbits are of the same type i.e. all the isotropy groups G_y are conjugated to a standard one, say, H.

2.2. Examples

a) Consider the natural action of the rotation group $SO(3)$ on $E = \mathbb{R}^3$. After removing the origin $0 \in \mathbb{R}^3$ we find that all the isotropy groups are conjugated to $SO(2)$ – the rotation group about z-axis. The space of orbits M is $\mathbb{R}^+$.

b) When we take $G = 0(2)$ acting on $E = \mathbb{R}^3$ we have to remove the z-axis to get one stratum. The isotropy groups are then all trivial. The space of orbits is the open half-plane $H = \mathbb{R}^+ \times \mathbb{R}$.

c) Take $E = SU(2) \simeq S^3$ and $G = U(1) \subset SU(2)$. Then all orbits are of the same type (all the stability groups are trivial) and $M = SU(2)/U(1) \simeq S^2$. Thus E is a fiber bundle with base S^2 and fiber S^1.

d) Let $E = U(2;H)$ and $G = U(1;H)$ acting on E by

$$U(1,H) \ni q : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} q^*aq & , q^*b \\ q^*cq & , q^*d \end{pmatrix}.$$

Here again all the stability groups are trivial, the space of orbits Σ^7 is topologically seven-sphere but carrying one of the exotic differentiable structures discovered by Milnor [16], [17].

2.3. Bundle structure of E

The triple (E, π, M) defined in 2.1 is a fibration but to make it into a fiber bundle one has to distinguish a class of local product representations (trivializations), and it is not obvious how to decide this question. We will see that (E, π, M) can be considered as an associated bundle with a typical fibre $H \backslash G$. The construction is considered standard in the mathematical literature (see e.g. [18]) but, so far, rarely used by physicists. One starts with observation that the isotropy groups along the orbit are mutually conjugated: $G_{ya} = a^{-1} G_y a$. Let $y \in E$ be arbitrary. By the assumption (iii) there exists $a \in G$ such that $G_y = aHa^{-1}$. But then $G_{ya} = H$. Thus on each orbit $x \in M$ the set P_x of all points having H as the isotropy group is non-empty. We have

$$P \equiv \{y \in E : G_y = H\} = \bigcup_{x \in M} P_x,$$

and $\pi : E \to M$ restricts to a projection $\pi : P \to M$. What is the structure of the fibers P_x of P? To answer this question let $y, y' \in P_x$. Since y and y' are on the same orbit there is an $a \in G$ such that $y' = ya$.

But then $H = G_y = G_{ya} = a^{-1} G_y a = a^{-1} Ha$. The set of all $a \in G$ with this property is known as the *normalizer* $N(H)$ of H in G:

$$N(H) = \{a \in G : aH = Ha\},$$

and is the biggest subgroup of G in which H is normal. Thus any two points of a fiber of P are related by an element of $N(H)$. The normalizer $N(H)$ acts on the fibers of P transitively, but the action is not effective – the subgroup H of $N(H)$ acts trivially on P. Since H is normal in $N(H)$, the quotient $K \equiv N(H) \mid H$ is a group. The action of K on the fibers of P is transitive and free. To conclude from this that (P, π, M, K) is a principal bundle, with structure group K, one has to know that P admits local cross-section. Existence of such cross-sections follows from the so-called «slice theorem» (see [18] and references there).

2.4. $\mathbf{K = N(H) \mid H}$ as the automorphism group of $\mathbf{H \setminus G}$

A geometrical structure of the homogeneous space $H \setminus G$ is determined by the action of G on it. Thus it is natural to define an *automorphism* of $H \setminus G$ as a map $\alpha : H \setminus G \to H \setminus G$ which *commutes* with this action

$$\alpha(za) = \alpha(z)a, \quad z \in H \setminus G, \quad a \in G.$$

The set of all automorphisms is a group under the composition. Let us show that this group is isomorphic with K. First of all, given $n \in N(H)$ define α_n by

$$\alpha_n([a]) \equiv [na].$$

If $[a] = [a']$ then $a' = ha$ and $na' = nha = nhn^{-1} na = h'na$. Thus $[a] = [a']$ implies $[na] = [na']$ and so the map α_n is well defined. We also have $\alpha_n([a]b) = \alpha_n([ab]) = [nab] = [na]b = \alpha_n(a)b$, therefore α_n is an automorphism. If $n' = hn$ then $\alpha_{n'}([a]) = [n'a] = [na] = \alpha_n([a])$, and, conversely, if $\alpha_{n'} = \alpha_n$ then $[n'] = \alpha_{n'}([e]) = \alpha_n([e]) = [n]$. Thus $n \to \alpha_n$ factorizes through H to a $1-1$ map $[n] \to \alpha_n$. It remains to show that every automorphism α is of this form. Given α let $n \in \alpha([e])$. It is straightforward to show that $n \in N(H)$ and $\alpha = \alpha_n$ what completes the proof.

REMARK. The following remark is important for avoiding misunderstandings.

Of course a particular example of a homogeneous space $H \backslash G$ is the group G itself. It corresponds to $H = \{e\}$. But an automorphism of G as a homogeneous space is a different concept from that of an automorphism of G as a group. The first would require $\alpha(ab) = \alpha(a)b$ while the second $\alpha(ab) = \alpha(a)\alpha(b)$. And it is clear that $H = \{e\}$ implies $N(H) = G$ and thus $K = G$ while the group of all automorphisms of the group G may be different from G (for example $\mathrm{Aut}\,(U(1)) = Z$).

2.5. Local product representations of E

Let $\sigma : M \to P$ be a local cross-section of the principal bundle P. Then σ determines a local trivialization $\phi : M \times K \to P$ of P by $\phi(x, [a]) \equiv \sigma(x)\, a$, $[a] \in N(H) \,|\, H$. But ϕ can be naturally extended to a map $\phi : M \times (H \backslash G) \to E$, and it is easy to see that ϕ so extended is a local diffeomorphism.

REMARKS.

a) we use the word «local» to indicate that M is to be understood as an open $U \subset M$, E as $\pi^{-1}(U)$ etc.
b) $H \backslash G$ denotes the space of *right* cosets $[a] = Ha$ on which G acts from the *right*. We write $N(H) \,|\, H$ with a vertical bar to indicate that, since H is normal in $N(H)$, left and right cosets of H in $N(H)$ coincide.
c) E may be thought of as a bundle associated to P via the left action of K on $H \backslash G : [n] : [a] \to [na]$, $[n] \in N(H) \,|\, H$. It is well known that a local cross-section of a principal bundle determines local product representation of every associated bundle. The map ϕ above is a particular example of such a representation.
d) In good texts on fiber bundles (see e.g. [19, Ch. 16.14.7.2]) there is a warning that the structure group does not act on associated bundles. One may wonder how to reconcile this with the remark c) and action of G on E. The crucial point here is that the *left* action of the structure group K and the *right* action of the group G on $H \backslash G$ *commute*.

3. RIEMANNIAN GEOMETRY OF H \ G

3.1. Lie algebra decomposition

G is a compact Lie group and H is a closed subgroup of G. The Lie algebras of G and H are $\mathcal{G}$ and $\mathcal{H}$ respectively. For technical reasons we assume H connected. Then Ad (H) invariance is the same as ad $(\mathcal{H})$-invariance. If one meets a case of H consisting of several components, then one can replace $H \backslash G$ by its covering to reduce H to the connected component of the identity. Recall that $N(H)$ is the normalizer of H in G and let $\mathcal{N}$ be the Lie algebra of $N(H)$. Consider now the following decompositions

– Let $G = N + L$, with $\mathrm{Ad}(N)\, L \subset L$,

– Let $N = H + K$, with $\mathrm{Ad}(H)\, K \subset K$,

– define $S \equiv K + L$.

Then

$$[H, K] = 0,$$

$$[K, K] \subset K,$$

$$\mathrm{Ad}(H) S \subset S,$$

S can be identified with the space tangent to $H \setminus G$ at the origin, and K can be identified with the Lie algebra of $K = N(H) | H$. Moreover, we also have (for a proof see [15])

$$K = \{\xi \in S : \mathrm{Ad}(h)\, \xi = \xi, h \in H\}. \tag{3.1.1}$$

The last property is very important. It tells us that K and L are orthogonal to each other with respect to *any* Ad (H) invariant scalar product on S.

3.2. The canonical moving frame

To each element $v \in G$ there corresponds a vector field Z_v on $H \setminus G$ - the fundamental vector field generated by v. It is defined by

$$Z_v(y) = \frac{\mathrm{d}}{\mathrm{d}t} (y e^{tv}) \Big|_{t=0} .$$

It follows from this definition that

$$Z_v a = Z_{\mathrm{Ad}(a^{-1})v}. \tag{3.2.1}$$

(we write $Z_v a$ for $(R_a)_* Z_v$) what implies

$$[Z_v, Z_w] = Z_{[v, w]}.$$

Let (ϵ_i) be a basis for the Lie algebra G. The basis is assumed to be adapted to the decomposition $G = H + K + L$, with $\epsilon_i = (\epsilon_{\underline{\alpha}}, \epsilon_\alpha)$, $\epsilon_{\underline{\alpha}} \in H$, $\epsilon_\alpha \in S$. The fundamental vector fields on $H \setminus G$ corresponding to ϵ_i are denoted by e_i. Then

$$[e_i, e_j] = C_{ij}^k\, e_k ,$$

where C_{ij}^k are the structure constants of G. At the origin $o = [e]$ the vector fields $e_{\underline{\alpha}}$ corresponding to the isotropy group H all vanish and $e_\alpha(0)$ form a basis in $T_o(H \setminus G)$. Thus e_α are linearly independent also in some open neighborhood of the origin. We call (e_α) the *canonical moving frame* for $H \setminus G$. It should be observed, however, that e_{α} may cease to vanish for y arbitrarily close to the origin (except for special directions generated by the action of $N(H)$).

3.3. Invariant metrics

If $g = (g_{\alpha\beta})$ is a G-invariant metric on $(H \setminus G)$ then its restriction to $T_0(H \setminus G) = S$ is Ad (H)-invariant. Conversely, every Ad (H)-invariant scalar product on S determines a G-invariant metric on $H \setminus G$, this owing to the transitivity of G-action. Now S decomposes into $K + L$ with $[H, K] = 0$, and $K \perp L$. Therefore to give $H \setminus G$ a G-invariant metric is to endow K and L with scalar products, the scalar product in L being Ad (H)-invariant and that in K arbitrary.

REMARK. We know from (2.4) that on $(H \setminus G)$ acts not only G from the right but also $N(H)$ from the left. There is, therefore, a subclass of G-invariant metrics on $(H \setminus G)$ consisting of metrics which are also $N(H)$-invariant. It is easy to see that these metrics are described by scalar product on S which are also Ad (K)-invariant. The K-part of such a scalar product determines a biinvariant metric on the group K. A particular example is given by the restriction to S of a biinvariant metric of G.

3.4. Curvature and Killing vectors

The fundamental vector fields Z_v, $v \in G$ are Killing vector fields for an invariant metric g. Thus $H \setminus G$ admits a moving frame of Killing vectors. We consider first a more general case of a metric g on space E admitting Killing vectors. If X, Y are vector fields, we denote by (X, Y) their scalar product given by the metric. Recall that X is a Killing vector if for all Y, Z

$$X(Y, Z) = ([X, Y], Z) + (Y, [X, Z]), \tag{3.4.1}$$

where $X(Y, Z)$ denotes the derivative of the function (Y, Z) in the direction of X.

Let ∇ be the Levi-Civita connection of $(\ ,\ )$ i.e.

$$X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z) \qquad \text{(metricity)} \tag{3.4.2}$$

and

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \qquad \text{(zero torsion)} \tag{3.4.3}$$

Decompose $\nabla_X Y$ into its symmetric and antisymmetric parts

$$\nabla_X Y = S(X, Y) + A(X, Y), \tag{3.4.4}$$

with $S(X, Y) = S(Y, X)$ and $A(X, Y) = -A(Y, X)$.

LEMMA 3.4.1. For any vector fields X, Y we have

$$A(X, Y) = \frac{1}{2}[X, Y]. \tag{3.4.5}$$

If X, Y, Z are Killing vectors then

$$(S(X, Y), Z) = -\frac{1}{2}\{([Z, X], Y) + (X, [Z, Y])\}. \tag{3.4.6}$$

Proof. The first statement follows from (3.4.3). To deduce the second observe that for any vector fields X, Y, Z one has

$$\begin{aligned}(\nabla_X Y, Z) &= \frac{1}{2}\{X(Y, Z) + Y(Z, X) - Z(X, Y)\} + \\ &+ \frac{1}{2}\{([X, Y], Z) - ([Y, Z], X) + ([Z, X], Y)\}.\end{aligned} \tag{3.4.7}$$

The result follows then using (3.4.1) for X, Y and Z. ∎

The curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

PROPOSITION 3.4.2. If X, Y, Z, W are Killing vectors, then

$$\begin{aligned}(R(X, Y)Z, W) = &-\frac{1}{4}\{([[X, Y], Z], W) - ([[X, Y], W]Z)\} + \\ &-\frac{1}{4}\{([[Z, W], X], Y) - ([[Z, W], Y], X)\} + \\ &+\frac{1}{4}\{([X, Z], [Y, W]) + 2([X, Y], [Z, W]) - ([Y, Z], [X, W])\} + \\ &+ (S(X, Z), S(Y, W)) - (S(Y, Z), S(X, W)).\end{aligned} \tag{3.4.8}$$

Proof. By (3.4.2) we have $(\nabla_X \nabla_Y Z, W) = X(\nabla_Y Z, W) - (\nabla_Y Z, \nabla_X W)$, and similarly for $(\nabla_Y \nabla_X Z, W)$. The formula follows then from (3.4.4) (3.4.5) and (3.4.7). ∎

3.5. Curvature of $H \setminus G$

We apply now Proposition 3.4.2 to derive a formula for the curvature of $H \setminus G$

endowed with an invariant metric g. For calculation we use the canonical moving frame e_α. With $X \to e_\alpha$, $Y \to e_\beta$, $Z \to e_\gamma$, (3.4.6) gives

(3.5.1) $$S_{\alpha\beta,\gamma} = -1/2\{C_{\gamma\alpha,\beta} + C_{\gamma\beta,\alpha}\},$$

where $C_{i\beta,\gamma} \equiv g_{\gamma\delta} C^\delta_{i\beta}$. (Observe that the structure constants $C_{i\alpha,\beta}$ are *not*, in general, antisymmetric with respect to the last two indices. This because g is *not* assumed to be *bi*invariant. However $C_{\underline{\alpha}\beta,\gamma} = -C_{\underline{\alpha}\gamma,\beta}$ owing to Ad (H)-invariance of $g_{\alpha\beta}$). From (3.4.5) and (3.5.1) we find the Christoffel symbols $\Gamma_{\alpha\beta,\gamma} = (\nabla_\alpha e_\beta, e_\gamma)$:

(3.5.2) $$\Gamma_{\alpha\beta,\gamma} = 1/2(C_{\alpha\beta,\gamma} - C_{\gamma\alpha,\beta} + C_{\beta\gamma,\alpha}).$$

Taking also $W \to e_\gamma$ and $R_{\alpha\beta\gamma\delta} = (R(e_\alpha, e_\beta)e_\gamma, e_\delta)$ we get

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{4}\{C^i_{\alpha\beta}C_{i\gamma,\delta} - C^i_{\alpha\beta}C_{i\delta,\gamma} + C^i_{\gamma\delta}C_{i\alpha,\beta} - C^i_{\gamma\delta}C_{i\beta,\alpha}\} +$$

(3.5.3) $$+\frac{1}{4}\{C^\kappa_{\alpha\gamma}C_{\beta\delta,\kappa} + 2C^\kappa_{\alpha\beta}C_{\gamma\delta,\kappa} - C^\kappa_{\beta\gamma}C_{\alpha\delta,\kappa}\} +$$

$$+\frac{1}{4}g^{\kappa\lambda}\{(C_{\kappa\alpha,\gamma} + C_{\kappa\gamma,\alpha})(C_{\lambda\beta,\delta} + C_{\lambda\delta,\beta}) - (C_{\kappa\beta,\gamma} + C_{\kappa\gamma,\beta})(C_{\lambda\alpha,\delta} + C_{\lambda\delta,\alpha})\}.$$

For the Ricci tensor $R_{\beta\gamma} = g^{\alpha\delta} R_{\alpha\beta\gamma\delta}$ one then gets

$$R_{\beta\gamma} = \frac{1}{4}C_{\alpha\kappa,\beta}C_{\alpha\kappa,\gamma} - \frac{1}{2}C_{\beta\alpha,\kappa}C_{\gamma\alpha,\kappa} - \frac{1}{2}C_{\beta\alpha,\kappa}C_{\gamma\kappa,\alpha} +$$

(3.5.4) $$-\frac{1}{2}C^{\underline{\kappa}}_{\beta\alpha}C^\alpha_{\gamma\underline{\kappa}} - \frac{1}{2}C^{\underline{\kappa}}_{\gamma\alpha}C^\alpha_{\beta\underline{\kappa}} +$$

$$-\frac{1}{2}(C_{\kappa\beta,\gamma} + C_{\kappa\gamma,\beta})C^\alpha_{\kappa\alpha},$$

where we use the convention that the summation over repeated indices on the same level is performed with $g^{\alpha\beta}$. Thus, for example

$$C_{\alpha\kappa,\beta}C_{\alpha\kappa,\gamma} = g^{\alpha\alpha'}g^{\kappa\kappa'}g_{\beta\beta'}g_{\gamma\gamma'}C^{\beta'}_{\alpha\kappa}C^{\gamma'}_{\alpha'\kappa'} \qquad \text{etc.}$$

For the scalar curvature $R = g^{\beta\gamma}R_{\beta\gamma}$ we obtain

(3.5.5) $$R = -\frac{1}{4}C_{\alpha\beta,\gamma}C_{\alpha\beta,\gamma} - \frac{1}{2}C_{\alpha\beta,\gamma}C_{\alpha\gamma,\beta} - C^{\underline{\kappa}}_{\beta\alpha}C^\alpha_{\beta\underline{\kappa}} - C^\alpha_{\kappa\alpha}C^\beta_{\kappa\beta}.$$

3.6. Comments

a) The last terms of the formulae (3.5.4) and (3.5.5) are included for the sake of completeness only. For a compact group, and more generally, for a unimodular group, we have (see e.g. [20, Ch. 19, 16 Prob. 14]) $C^i_{\alpha i} = 0$, and therefore (since $C^{\underline{h}}_{\alpha\underline{\kappa}} = 0$ by reductiveness of $G = H + S$) the last terms in (3.5.4) and (3.5.5) vanish. In the following these terms will be omitted.

b) $H \backslash G$ is called naturally reductive homogenous space if $C_{\alpha\beta,\gamma}$ are antisymmetric also in the last two indices. Then $\nabla_X Y = 1/2\,[X, Y]$ and

$$R_{\beta\gamma} = \frac{1}{4}\, C^{\alpha}_{\beta\kappa}\, C^{\kappa}_{\gamma\alpha} + \frac{1}{2}\, k_{\beta\gamma}\,,$$

$$R = \frac{1}{4}\, C^{\alpha}_{\beta\kappa}\, C^{\kappa}_{\beta\alpha} + \frac{1}{2}\, g^{\beta\gamma} k_{\beta\gamma}\,,$$

where $k_{ij} = -\, C^m_{il}\, C^l_{jm}$ is the Killing metric (nonnegative, minus the Killing form!)

c) In particular if $H \backslash G$ is a symmetric space (i.e.; $C_{\alpha\beta,\gamma} = 0$), then

$$R_{\beta\gamma} = \frac{1}{2}\, k_{\beta\gamma}\,,$$

$$R = \frac{1}{2} g^{\beta\gamma} k_{\beta\gamma}$$

d) $H \backslash G$ is called a normal space if $g_{\alpha\beta}$ is a restriction to $S \doteq G \ominus H$ of a biinvariant metric g_{ij} on G. Clearly d) implies b). Conversely, if G is connected then b) implies d) but g_{ij} may be, in general, semidefinite. The subparticular cases are

d$_1$) $H = \{e\}$ (the group case) and $g_{\alpha\beta} = k_{\alpha\beta}$. Then

$$R_{\beta\gamma} = \frac{1}{4}\, k_{\beta\gamma}\,,$$

$$R = \frac{1}{4} \dim G$$

d$_2$) $H \backslash G$ is a symmetric homogeneous space and $g_{\alpha\beta} = k_{\alpha\beta}$. Then

$$R_{\beta\gamma} = \frac{1}{2} k_{\beta\gamma},$$

$$R = \frac{1}{2} \dim H \setminus G.$$

e) The Ricci tensor $R_{\beta\gamma}$ considered as a symmetric bilinear form on S is also Ad (H)-invariant. If $H \setminus G$ is *isotropy irreducible* i.e. if Ad (H) acts irreducibly on S, then, by the Shur's Lemma, $R^{\alpha}_{\beta} \equiv g^{\alpha\gamma} R_{\beta\gamma}$ must be a multiple of δ^{α}_{β} and therefore $R_{\beta\gamma} = \lambda g_{\beta\gamma}$. Thus $H \setminus G$ is an *Einstein* space.

f) We considered the space of *right* cosets $H \setminus G$. For a *left* coset space G / H the fundamental vector fields satisfy $[Z_v, Z_w] = -Z_{[v,w]}$ with the effect that everywhere in our formulas C^k_{ij} is to be replaced by - C^k_{ij}. Such a change has no effect on the curvature formulae which are all quadratic in structure constants.

g) The formulae (3.5.3), (3.5.4) and (3.5.5) hold at the origin $o \in H \setminus G$ and, more generally, at any other point $p \in H \setminus G$ for which the isotropy group G_p is H. To go to an arbitrary point $[a]$ one has to transform the indices by the adjoint representation Ad (a).

4. G-INVARIANT DIMENSIONAL REDUCTION OF METRIC

4.1. Reduction theorem

E is not a homogeneous space, it is a collection of homogeneous spaces E_x parametrized by points $x \in M$. In this section we describe all metrics on E which are G-invariant. The simplest description is geometrical - without any formula!

REDUCTION THEOREM [15]. Every G-invariant metric on E determines, and is determined by, a triple consisting of

i) for each $x \in M$, a G-invariant metric in E_x - the copy of $H \setminus G$ over x

ii) principal connection in the principal bundle (P, π, M)

iii) metric on M. ■

Let us discuss briefly the three ingredients.

Ad i) Clearly G-invariant metric on E restricts to each fiber E_x, and determines a G-invariant metric on $H \setminus G$.

Ad ii) For each $y \in E$ let V_y be the subspace of the tangent space $T_y E$ consisting of vectors tangent to the orbits. Define H_y to be the orthogonal complement of V_y. Then H_y, restricted to $y = p \in P$, is a K-invariant horizontal distribution. To conclude that $(H_p)_{p \in P}$ determines a principal connection in P one has to show that H_p is tangent to P. This follows by using orthogonality of K and L (see end of Section 3.1).

Ad iii) Given pair of vectors $\xi, \eta \in T_x M$ we can lift them to $\hat{\xi}(y), \hat{\eta}(y) \in H_y$ for any $y \in E_x$. Then, because of G-invariance of the metric, we find that $(\hat{\xi}(y), \hat{\eta}(y))$ is *independent* of y and thus determines a scalar product at x. One can express this fact by saying that $\pi : E \to M$ is a Riemannian submersion.

4.2. The adapted moving frame

Each tangent space $T_y E$ decomposes into

$$T_y E = V_y \oplus H_y$$

where $V_y = \{Z_v(y) : v \in G\}$, called the vertical space at y, is the space tangent to the orbit of G through y, and H_y (called horizontal) is defined as the orthogonal complement of V_y in $T_y E$. The vectors in V_y and H_y are called vertical and horizontal respectively. We fix a local coordinate system x^μ in M and denote by e_μ the horizontal lifts of vector fields ∂_μ. The fundamental vector fields corresponding to the basic vectors $\epsilon_i \in G$ are denoted, as in Section 3, by e_i. Then $(e_A) = (e_\mu, e_\alpha)$ is a moving frame in a neighbourhood of a point $p_0 \in P$. The three ingredients of a G-invariant metric $g = (g_{AB}) = [g(e_A, e_B)]$ can be constructed now as follows

$$g_{\mu\nu}(x) = g(e_\mu(y), e_\nu(y)), \qquad \pi(y) = x, \qquad x \in M \tag{4.2.1}$$

$$g_{\alpha\beta}(p) = g(e_\alpha(p), e_\beta(p)), \qquad p \in P, \tag{4.2.2}$$

$$\omega^{\hat{a}}(v_p) = v^{\hat{a}}, \qquad v_p = v^{\hat{a}} e_{\hat{a}}(p) + v^\mu e_\mu(p) \in T_p P, \tag{4.2.3}$$

where $e_\alpha = (e_{\hat{a}}, e_a)$ corresponds to the decomposition $\mathscr{S} = \mathscr{K} + \mathscr{L}$, and $\omega^{\hat{a}}$ is the connection form determined by the horizontal distribution $(H_p)_{p \in P}$.

Let us now write down the commutation relations for e_A. We have

$$[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta} e_\gamma = C^i_{\alpha\beta} e_i, \qquad f^\gamma_{\alpha\beta}(p) = C^\gamma_{\alpha\beta} \quad \text{for} \quad p \in P \tag{4.2.4}$$

$$[e_\alpha, e_\mu] = 0 \tag{4.2.5}$$

$$[e_\mu, e_\nu] = -F^\alpha_{\mu\nu} e_\alpha . \tag{4.2.6}$$

Comments

a) The first relation is evident. Since e_α are linearly independent we must have $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta} e_\gamma$ for some structure *functions* $f^\gamma_{\alpha\beta}$. On the other hand $[e_\alpha, e_\beta] = C^i_{\alpha\beta} e_i$ since e_α are fundamental. On P we have $e_{\underline{\alpha}} \equiv 0$ and therefore $f^\gamma_{\alpha\beta}$ are *constant* on P.

b) The fields e_μ are *invariant* by construction, therefore (4.2.5) holds.

c) Because $[\partial_\mu, \partial_\nu] = 0$ it follows that $[e_\mu, e_\nu]$ is purely vertical. Thus (4.2.6) defines $F^\alpha_{\mu\nu}$. However, at $p \in P$, the vectors e_μ, e_ν are *tangent* to P. It follows that also $[e_\mu, e_\nu](p)$ is tangent to P. Therefore $F^a_{\mu\nu}(p) = 0$, and $F^{\hat{a}}_{\mu\nu}$ is the curvature 2 - form of ω.

d) For $n \in N$ we have $pn \in P$ for $p \in P$, and

$$e_\alpha(pn) = A\,(n^{-1})^{\alpha'}_\alpha e_{\alpha'}(p), \tag{4.2.7}$$

$$g_{\alpha\beta}(pn) = A\,(n^{-1})^{\alpha'}_\alpha A\,(n^{-1})^{\beta'}_\beta g_{\alpha'\beta'}(p), \tag{4.2.8}$$

where $A\,(n)^{\alpha'}_\alpha$ is the matrix of the adjoint representation

$$\mathrm{Ad}\,(n)\epsilon_\alpha = A\,(n)^{\alpha'}_\alpha \epsilon_{\alpha'}.$$

The scalar fields $g_{\alpha\beta}$ satisfy the constraint of Ad (H)-invariance, infinitesimaly

$$g_{\alpha\gamma}(p)\, C^\gamma_{\underline{\alpha}\beta} + g_{\beta\gamma} C^\gamma_{\underline{\alpha}\alpha} = 0,$$

which owing to the assumed connectedness of H, is also sufficient for Ad (H) invariance. According to (4.2.8) $g_{\alpha\beta}$ depends on $p \in P$ in a covariant way. Thus it can be interpreted as a section of an associated bundle.

4.3. The Levi-Civita connection

The structure functions of the moving frame e_A are given by (4.2.4)-(4.2.6). From G-invariance we also have, at $p \in P$,

$$e_\mu(e_\alpha, e_\beta) = D_\mu g_{\alpha\beta}, \tag{4.3.1}$$

$$e_\alpha(e_\beta, e_\gamma) = C_{\alpha\beta,\gamma} + C_{\alpha\gamma,\beta}. \tag{4.3.2}$$

We use now the formula (3.4.8) to express the Christoffel symbols $\Gamma_{AB,C} = (\nabla_A e_B, e_C)$ in terms of M-based quantities:

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\,(f_{\alpha\beta,\gamma} - f_{\gamma\alpha,\beta} + f_{\beta\gamma,\alpha}) \tag{4.3.3}$$

$$\Gamma_{\mu\alpha,\beta} = \Gamma_{\alpha\mu,\beta} = -\,\Gamma_{\alpha\beta,\mu} = \frac{1}{2} D_\mu g_{\alpha\beta} \tag{4.3.4}$$

$$\Gamma_{\mu\nu,\alpha} = -\,\Gamma_{\mu\alpha,\nu} = -\,\Gamma_{\alpha\mu,\nu} = -\frac{1}{2} F_{\mu\nu,\alpha} \tag{4.3.5}$$

$\Gamma_{\mu\nu,\sigma} = \{$the Christoffel symbols of $g_{\mu\nu}$ on $M\}$.

Comments

a) $D_\mu g_{\alpha\beta}$ denotes the covariant derivative of $g_{\alpha\beta}$ with respect to the connection ω. With respect to a local cross section $\sigma : M \to P$ it can be explicitly written as

$$D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} + C^\gamma_{\alpha\hat{a}} A^{\hat{a}}_\mu g_{\gamma\beta} + C^\gamma_{\beta\hat{a}} A^{\hat{a}}_\mu g_{\alpha\gamma} \,,$$

where $A^{\hat{a}} = \sigma^* \omega^{\hat{a}}$ is the Yang-Mills potential. Strictly speaking (4.3.1), (4.3.4) and (4.3.5) hold on P only. On the other hand (4.3.1) can be considered as a *definition* of $D_\mu g_{\alpha\beta}$ outside of P. This is similar to the interpretation we have given to (4.2.6).

b) The structure functions $f^\gamma_{\alpha\beta}$ are, according to (4.2.4) constant on P. Thus (4.3.3) is nothing but (3.5.2).

4.4. Ricci and scalar curvature

We give below the formulae for Ricci and scalar curvature of E. In the adapted moving frame (e_A) defined in 4.2 we obtain

$$R_{\alpha\beta} = R_{\alpha\beta}(H \setminus G) + \frac{1}{4} F_{\mu\nu,\alpha} F_{\mu\nu,\beta} + \frac{1}{2} g^{\gamma\delta} D_\mu g_{\alpha\gamma} D^\mu g_{\beta\delta} - \frac{1}{4} D^\mu g_{\alpha\beta} g^{\gamma\delta} D_\mu g_{\gamma\delta} - \frac{1}{2} \mathcal{D}_\mu (D^\mu g_{\alpha\beta}) \tag{4.4.1}$$

$$R_{\mu\nu} = R_{\mu\nu}(M) - \frac{1}{2} F_{\mu\sigma,\alpha} F_{\nu\sigma,\alpha} - \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} D_\mu g_{\alpha\gamma} D_\nu g_{\beta\delta} - \frac{1}{2} \mathcal{D}_\mu (g^{\alpha\beta} D_\nu g_{\alpha\rho} \tag{4.4.2}$$

$$R_{\mu\alpha} = \frac{1}{2} \mathcal{D}^\sigma F_{\mu\sigma,\alpha} + \frac{1}{4} F_{\mu\sigma,\alpha} g^{\beta\gamma} D^\sigma g_{\beta\gamma} - \frac{1}{2} C^\gamma_{\alpha\beta} g^{\beta\delta} D_\mu g_{\gamma\delta} \tag{4.4.3}$$

$$R = R(M) + R(H \setminus G) - \frac{1}{4} F_{\mu\nu,\alpha} F_{\mu\nu,\alpha} - \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} (D_\mu g_{\alpha\gamma} D^\mu g_{\beta\delta} + D_\mu g_{\alpha\beta} D^\mu g_{\gamma\delta}) - \nabla_\mu (g^{\alpha\beta} D^\mu g_{\alpha\beta}). \tag{4.4.4}$$

Comments

a) $R_{\alpha\beta}(H \setminus G)$ and $R(H \setminus G)$ are given (3.5.4) and (3.5.5) with understanding that $g_{\alpha\beta}$ is a function of x.

b) The derivative $\mathcal{D}_\mu$ in (4.4.1) and (4.4.2) acts both on internal *and* space-time indices with $A^{\hat{a}}_\mu$ and $\Gamma^\alpha_{\mu\nu}$ respectively. The derivative ∇_μ in (4.4.4) is the space-time covariant derivative. It can also be understood as $\mathcal{D}_\mu$.

c) Let us give an example of ca lations used in derivation of (4.4.4):

$$g^{\alpha\beta}R_{\alpha\beta} = \ldots - \frac{1}{2} g^{\alpha\beta}\mathcal{D}_\mu(D^\mu g_{\alpha\beta}) = \ldots - \frac{1}{2}\mathcal{D}_\mu(g^{\alpha\beta}D^\mu g_{\alpha\beta}) + \frac{1}{2}D_\mu g^{\alpha\beta}D^\mu g_{\alpha\beta} =$$

$$= \ldots - \frac{1}{2}\nabla_\mu(g^{\alpha\beta}D^\mu g_{\alpha\beta}) - \frac{1}{2}g^{\alpha\gamma}D_\mu g_{\gamma\lambda}g^{\lambda\beta}D^\mu g_{\alpha\beta}.$$

d) The summation over repeated indices on the same level is performed with $g^{\alpha\beta}$ and $g^{\mu\nu}$. For example, at $p \in P$ the term $F_{\mu\nu,\alpha}F_{\mu\nu,\alpha}$ should be read as $g^{\mu\mu'}g^{\nu\nu'}g_{\hat{a}\hat{a}'}F^{\hat{a}}_{\mu\nu}F^{\hat{a}'}_{\mu'\nu'}$.

e) The fourth term in (4.4.4), with the derivatives $D_\mu g_{\alpha\beta}$, is printed in [15] with the wrong factor $\frac{1}{2}$ instead of $\frac{1}{4}$. The preprint (CERN) version of [15] gives the correct factor.

f) There are several possibilities of using (4.4.1) - (4.4.4) for determining field equations.

   i) According to the extrem Kaluza-Klein philosophy the field equations are $R_{AB} = 0$ (a ming no matter sources in E). This means $R_{\mu\nu} = R_{\alpha\beta} = R_{\mu\alpha} = 0$. is possible that the extreme philosophy should be used s. ltaneously with the complete harmonic analysis of excitations from the ground state and need not be compatible with an ad hoc restriction to G-invariant modes.

   ii) One can also try to get a dynamics of G-invariant modes from the action principle on M. The n ural candidate for the action is (see e.g. [14])

$$S \sim \int_M R(g_{\mu\nu})^{\frac{1}{2}}(g_{\alpha\beta})^{\frac{1}{2}}\mathrm{d}x$$

The last term of (4.4.4) gives then

$$-\nabla_\mu(g^{\alpha\beta}D^\mu g_{\alpha\beta})(g_{\mu\nu})^{\frac{1}{2}}(g_{\alpha\beta})^{\frac{1}{2}} = -\mathcal{D}_\mu(g^{\alpha\beta}D^\mu g_{\alpha\beta}(g_{\mu\nu})^{\frac{1}{2}}(g_{\alpha\beta})^{\frac{1}{2}}) +$$

$$+\mathcal{D}_\mu(g_{\alpha\beta})^{\frac{1}{2}}(g^{\alpha\beta}D^\mu g_{\alpha\beta})(g_{\mu\nu})^{\frac{1}{2}} = -\partial_\mu(g^{\alpha\beta}D^\mu g_{\alpha\beta}(g_{\mu\nu})^{\frac{1}{2}}(g_{\alpha\beta})^{\frac{1}{2}}) +$$

$$+\frac{1}{2}g^{\gamma\delta}D_\mu g_{\gamma\delta}g^{\alpha\beta}D^\mu g_{\alpha\beta}(g_{\alpha\beta})^{\frac{1}{2}}(g_{\mu\nu})^{\frac{1}{2}}.$$

The term $\partial_\mu(\ldots)$ does not contribute to field equations. For calculation of $\det(g_{\alpha\beta})^{1/2}$ any fixed basis (ϵ_α) is S can be used. It is, however, convenient to use (ϵ_α) which is orthonormal for a normal metric $\mathring{g}$ normalized to give a unit volume of $H \setminus G$. With this choice $\det(g_{\alpha\beta}(x))^{1/2}$ *is* the volume of E_X, and integration over the fiber is already performed.

iii) Another possibility is to take $S \sim \int_M R(g_{\mu\nu})^{1/2} \mathrm{d}x$. (e.g. [21]). With this choice the last term of (4.4.4) is already a divergence. However, such a choice is considered as too arbitrary and too eclectic: either we take extra dimensions seriously, or if not, then why to bring them in at all?

vi) Finally, one can make a conformal transformation

$$g_{\mu\nu} \to [\det(g_{\alpha\beta})]^r g_{\mu\nu}$$

where r is chosen in such a way that ii) $\to$ iii) + (terms with $\partial_\mu g_{\alpha\beta}$). (e.g. [22]).

5. SYMMETRIES OF GAUGE FIELDS

5.1. Example

It is instructive to apply the methods developed above to a particular case considered by Cho. Cho [23] considers Riemannian metrics on a principal bundle which are invariant not only with respect to the structure group acting from the right, but also with respect to an extra symmetry group (Cho calls it «magnetic») acting on the bundle space from the left. Let (E, π, M, R) be the principal bundle, R its structure group, and let S be a group of *inner* automorphisms (or gauge transformations) of E. Every $s \in S$ maps E_x into E_x and commutes with R: $(sy)a = s(ya)$, $a \in R$. The full symmetry group is now $G = S \times R$. It is this group which acts on E now, and to apply the machinery we were developing we have to know the isotropy group H. Fix $y \in E$ and let $\lambda : S \to R$ be the group homomorphism defined by

$$sy = y\lambda(s), \qquad s \in S.$$

Then the isotropy group $G_y = H$ of y is

$$H = \{(s, \lambda(s)) : s \in S\}. \tag{5.1.1}$$

REMARK. H is the isotropy group for the *right* action of G, where the right action of S is *defined* by $ys \equiv s^{-1}y$.

Suppose that all orbits are of the same type. Then we know that $P \equiv \{y \in E : G_y = H\}$

is a principal bundle with structure group $N(H)\,|\,H$. Let us find $N(H)$ and $N(H)\,|\,H$. We have $(\sigma,\rho)\in N(H)$ iff $(\sigma,\rho)\,(s,\ \lambda(s))\,(\sigma^{-1},\rho^{-1})=(s',\lambda(s'))$. Thus $s'=\sigma s\sigma^{-1}$ and $\rho\lambda(s)\rho^{-1}=\lambda(s')=\lambda(\sigma)\lambda(s)\lambda(\sigma)^{-1}$. It follows that $c\equiv\lambda(\sigma^{-1})\rho\in\lambda(S)'$ - the centralizer (commutant) of $\lambda(S)$ in R. Therefore $N(H)=\{(s,\lambda(s)\,c):s\in S,$ $c\in\lambda(S)'\}$, and $N(H)\,|\,H=\lambda(S)'$. The effective gauge group is the centralizer of the λ-image of S in R.

5.2. Group action on a principal bundle.

We now generalize the previous example to include also automorphisms which are not inner. Let (U,π,E,R) be a principal bundle with structure group R, and let S be a (compact Lie) group of *automorphisms* of U. Now S acts both on U *and* $E:s:U_x\to U_{xs}$ and commutes with the principal action of R:

$$(ys)r=(s^{-1}y)r=s^{-1}(yr)=(yr)s. \tag{5.2.1}$$

Of course S may contain a nontrivial subgroup S° of *inner* automorphisms as discussed above. In such a case the action of S on E is not *effective*. The full symmetry group acting on U is now $G=S\times R$. Let us find out how the isotropy groups of the G-action look like. Fix $u\in U$, and let $H=G_u$ be the isotropy group of u. It is clear that $(s,r)\in H$ implies that s is in the isotropy group, call it I, of $\pi(u)=y$. Now, if $s\in I$ then su is in the same fiber as u. It follows that $su=u\lambda(s)$, for some $\lambda(s)\in R$. The map $\lambda:I\to R$ is a group homomorphism. And $su=u\lambda(s)$ can be also read as

$$H=\{(s,\lambda(s)):s\in I\} \tag{5.2.2}$$

H, the isotropy group of $u\in U$, is a subgroup of $G\times R$ *isomorphic* to I - the isotropy group of $y=\pi(u)$. But it is imbedded in $G\times R$ on a diagonal. We shall assume that $G\times R$ acts on U with only *one orbit type*, that is that the isotropy group of any $u\in U$ is conjugated to H. (It implies that the isotropy group of any $y\in E$ is conjugated to I, but the inverse need not be true).

Consider now the manifold $M=U/G$ of orbits of $G=S\times R$ in U. Since U is locally $E\times R$ it follows that $U/S\times R$ is the same as E/S:

$$M=U/G=E/S. \tag{5.2.3}$$

We can introduce now two principal bundles

$$\begin{aligned} Q&=\{u\in U:G_u=H\},\\ P&=\{y\in E:S_y=I\}, \end{aligned} \tag{5.2.4}$$

with structure groups $N(H)\,|\,H$ and $K\equiv N(I)\,|\,I$ respectively. Now, if $u\in Q$ and $s\in I$, then $su=u\lambda(s)\Rightarrow\pi(su)=\pi(u)\Rightarrow s\pi(u)=\pi(u)\Rightarrow\pi(u)\in P$. It follows that

$\pi : U \to E$ restricts to a map $\pi : Q \to P$. We have therefore the diagram

$$\begin{array}{ccccc} U & & \to & & E \\ & \searrow & & \swarrow & \\ \uparrow & & M & & \uparrow \\ & \nearrow & & \nwarrow & \\ Q & & \to & & P. \end{array}$$

However the map $\pi : Q \to P$ need not be *onto* P if K is not connected. Let us see what is the local structure of the group $N(H)|H$. First of all we find, with $s \in I$,

$$(\sigma, \rho) \in N(H) \Leftrightarrow (\sigma, \rho)(s, \lambda(s))(\sigma, \rho)^{-1} = (s', \lambda(s')) \Leftrightarrow (\sigma s \sigma^{-1} = s', \rho\lambda(s)\rho^{-1} = \lambda(s'))$$

that is

$$(\sigma, \rho) \in N(H) \Leftrightarrow \begin{cases} \sigma \in N(I) \\ \\ \rho\lambda(s)\rho^{-1} = \lambda(\sigma s \sigma^{-1}) \qquad s \in I. \end{cases}$$

We know from Section 3.1 that $N(I)$ is locally a product $N(I) \overset{\text{loc}}{=} I \times K$. Indeed, the Lie algebra of $N(I)$ is the direct sum of the two commuting subalgebras. Now, if $\sigma = ik$ with $i \in I$ and $k \in K$, then, with $s \in I$,

$$\lambda(\sigma s \sigma^{-1}) = \lambda(iksk^{-1}i^{-1}) = \lambda(isi^{-1}) = \lambda(i)\lambda(s)\lambda(i)^{-1},$$

and therefore $\lambda(i)^{-1}\rho \in Z \equiv \lambda(I)'$, and

$$\rho = \lambda(i)z, \qquad z \in Z.$$

Thus, locally, we have

$$N(H) \overset{\text{loc}}{=} \{(ik, \lambda(i)z)\},$$

and therefore

$$N(H)|H \overset{\text{loc}}{=} K \times Z.$$

On a global level it is easy to see that Z is an invariant subgroup of $N(H)|H$ and $(N(H)|H)|Z$ is naturally imbedded into K, but, in general, the natural map of $N|H \times Z$ into K (defined by a factorization of $pr_1 : (s, r) \to s$) need not give all the connected components of K.

REMARKS

a) Z is an invariant subgroup of $N(H)|H$, the structure group of Q, but Q need not admit a reduction to Z. In fact, since $Q/Z \subset P$, if Q can be reduced to Z, then P is trivial.

b) The first papers dealing with the problem of symmetric gauge fields [25, 26, 27] concentrated on the centralizer Z and overlooked the normalizer, i.e. $N(I)\,|\,I$ factor.

c) In [26] the authors had to solve the problem of finding a local cross-section σ of U with fixed homomorphism λ. In our, geometrical, language σ is a cross-section of the principal bundle Q, and its existence is guaranted by the «slice theorem» mentioned in Section 2.3 (1).

5.3. Inner symmetries

(U, π, E, R) is a principal bundle and let ω be a principal connection. If $s : U \to U$ is an automorphism of U then $s^*\omega$ is again a connection form. It is natural to call s a symmetry of ω if $s^*\omega$ differs from ω only by a gauge transformation i.e. by an inner automorphism τ_s

(5.3.1) $$s^*\omega = \tau_s^*\omega.$$

Now if s and s' are symmetries, then

$$(ss')^*\omega = (\tau_{ss'})^*\omega$$

and $(ss')^*\omega = s'^*(s^*\omega) = s'^*(\tau_s^*\omega) = s'^*\tau_s^* s'^{-1*}\,\tau_{s'}^*\,\omega.$

It follows that $\tau_{ss'}$ may differ from $\tau_{s'}s'^{-1}\tau_s \cdot s'$ by an inner automorphism which leaves ω invariant. It is important to know this group. We call it $J(\omega)$.

PROPOSITION 5.3. Suppose E connected. Then $J(\omega)$ is isomorphic to the centralizer of the holonomy group of ω.

Proof. For $u, v \in U$ define $u \sim v$ iff u and v can be joined by a horizontal (piecewise differentiable) path in U. If α is an inner automorphism of U which leaves ω invariant then $u \sim v$ iff $\alpha(u) \sim \alpha(v)$. Let $\Phi(u)$ be the holonomy group at u:

$$\Phi(u) = \{a \in R : u \sim ua\}.$$

Fix $u \in U$, and let $\lambda_u : J(\omega) \to R$ be defined by

$$\alpha(u) = u\lambda_u(\alpha), \qquad \alpha \in J(\omega).$$

It is straightforward to check that λ_u is $1-1$ and that it maps $J(\omega)$ onto the centralizer of $\Phi(u)$ in R. For example, let us see that λ_u is *onto*. Let $b \in \Phi(u)'$. To define α take any $v \in U$ and let γ be a horizontal path connecting v with the fiber through u i.e. with ua, for some $a \in R$. Then define $\alpha(v) \equiv va^{-1}ba$. It is

(1) I am indebted to Dr. J. Tafel for pointing this question out.

straightforward to see that α does not depend on the choice of γ, that $\alpha \in J(\omega)$, and that $b = \lambda_u(\alpha)$, what completes the proof. ∎

The group $J(\omega)$ is of utmost important when one consider the problem of *lifting* of symmetries from the base to the bundle. $J(\omega)$ describes the freedom of choosing a «phase factor» of the transformation and, in general, a symmetry group after lifting to the bundle space will acquire a *multiplier* with values in $J(\omega)$. This phenomenon of appearing of projective representations of groups is well known from quantum mechanics. The $U(1)$ case is, however, exceptionally simple. Indeed the centralizer of the homomony group is in this case *always* $U(1)$, independently of the connection. For a non-Abelian gauge field the freedom of choosing a phase will depend on the gauge field.

5.4. Killing vectors of a connection

The formula (5.3.1) tells us when a *given* automorphism of the bundle can be considered as a symmetry of ω. It can be read as $s'^*\omega = \omega$, with $s' = \tau_s^{-1}s$. Thus for a given ω the important group is the group of all automorphisms of U which leave ω invariant. Infinitesimal automorphisms are described by *invariant* vector fields. A vector field X on the bundle is invariant if $Xa = X$ for all $a \in R$ or, infinitesimally, if

$$(5.4.1) \qquad [X, Z_v] = 0, \qquad v \in \mathcal{R},$$

where Z_v is the fundamental vector field generated by element v of $\mathcal{R} = \text{Lie}\,(R)$. An invariant vector field X will be called a Killing vector for ω if

$$(5.4.2) \qquad L_X \omega = 0$$

i.e.

$$(5.4.3) \qquad X\omega(Y) - \omega([X, Y]) = 0.$$

The curvature 2-form $F \equiv D\omega$ is defined by

$$(5.4.4) \qquad F(Y,Z) = (D\omega)(Y,Z) = Y\omega(Z) - Z\omega(Y) - \omega([Y,Z]) + [\omega(Y), \omega(Z)].$$

Combining (5.4.3) and (5.4.4) we find that *if X, Y are two Killing vectors* of ω then

$$(5.4.5) \qquad F(X, Y) = \omega([X, Y]) + [\omega(X), \omega(Y)].$$

This formula is similar in its content to (3.4.9) where we have given matrix elements of the curvature tensor of a metric between Killing vectors.

Every Killing vector of ω can be decomposed into its vertical and horizontal part $X = X_v + X_h$, with both X_v and X_h being again Killing vectors. We already know that X_v span the Lie algebra of the centralizer of the holonomy group of ω. It is instructive to understand this property also algebraically. First of all observe that a vertical invariant vector field χ can be also interpreted as a matter field of type Ad i.e. as a cross section of the associated vector bundle $U \times_{\mathrm{Ad}(R)} R$. Indeed, vertical vectors can be identified with vectors in the Lie algebra and invariance of χ means that $\chi(pa) = \mathrm{Ad}\,(a^{-1})\chi(p)$ because of the property (3.2.1) of fundamental fields. With this double understanding of χ one can easily find that

(5.4.6) $$L_\chi \omega = D\chi.$$

Thus χ *is a Killing vector for* ω iff χ *is covariantly constant*. Suppose now $L_\chi \omega = D\chi = 0$, Then, because of the identity

(5.4.7) $$D^2\chi = [F, \chi]$$

we find that the values of any vertical Killing field must commute with F what is compatible with the Proposition (5.3).

Let us consider horizontal Killing vectors. Every invariant horizontal vector field (X is horizontal iff $\omega(X) \equiv 0$) is a horizontal lift of a vector field on the basis. If ξ is a vector field on E let $\hat{\xi}$ be its horizontal lift. Then $\hat{\xi}$ is a Killing vector of ω iff

(5.4.8) $$i(\xi)F = 0,$$

i.e.

(5.4.9) $$F(\xi, \eta) = 0 \qquad \forall \eta.$$

5.5. Example

It is instructive to discuss a simple example given by Henneaux [28] (2). He considers a gauge potential described in Minkowski space by $\omega_\mu(x) = \delta^0_\mu x^1 M$, where $M \neq 0$ is a constant matrix from the $SO(3)$ algebra, and argues that translations when lifted to the bundle must acquire phase factors which lead to a nontrivial multiplier. Let us briefly comment on this example. The holonomy group of ω_μ is $U(1)$ and its centralizer in $SO(3)$ is again $U(1)$. Thus $J(\omega) = U(1)$ and we *may* have $U(1)$ multiplier. To simplify further reasoning suppose the gauge group was $U(1)$ instead of $SO(3)$. Since $F_{01}(x) \equiv M \neq 0$ and the gauge group is Abelian, it follows from the formula (5.4.5) that infinite-

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(2) I am indebted to Prof. A. Trautman for drawing my attention to this reference.

simal translations along the x^0 and x^1 axis *must* lift to *noncommuting* Killing vectors X_0 and X_1. The fact that the gauge group is $SO(3)$ and not $U(1)$ does not change this conclusion since a moment of reflection shows that $\omega(X_0)$ and $\omega(X_1)$ must commute also in this case.

6. DIMENSIONAL REDUCTION OF EINSTEIN-YANG-MILLS SYSTEMS

6.1. Application of the Reduction Theorem

We come back to the problem of describing all Yang-Mills fields with a given symmetry group S. As in section (5.1) we assume that S acts already on the bundle and that connections are strictly invariant i.e. the τ_s in (5.3.1) is put on the left-hand side. Our discussion in Section (5.2) indicates that a G-invariant connection ω may induce a connection in the bundle Q defined in (5.2.4). It was shown in [24] that the situation is more complicated. ω supplies only Z--part of a connection in Q and a $N(I)|I$-part must be supplied by S-invariant metric on E. Therefore a natural object for dimensional reduction is not a Yang--Mills field but an Einstein-Yang-Mills system. Another justification to this assertion is that the Yang-Mills Lagrangian for a connection ω_U on U involves metric g_E on E, and if the action is to be G-invariant then *not only* ω_U *but also* g_E *must be* G-*invariant*.

Now, let ω_U and g_E be both G-invariant. According to the Reduction Theorem given in Section (4.1) we can use a fixed biinvariant metric δ_R on the structure group R to built an R-invariant metric g_U on U. Since the ingredients g_U was built from were all S-invariant, it follows that g_U is not only R - but also $R \times S$--invariant. Thus the problem of classifying all S-invariant Einstein-Yang-Mills systems has been reduced to the problem of classifying all $R \times S$ invariant metrics which induce a fixed biinvariant metric on R. And this latter problem can be easily handled with the methods we already learned. According to the Reduction Theorem we have (observe that $U/S \times R = M$)

$$g_U \Leftrightarrow \begin{cases} g_M \\ \omega_Q \\ \text{scalar fields} \end{cases}$$

The g_M and ω_Q parts are clear. Let us discuss the scalar fields. To know their nature we must decompose the Lie algebra $\mathcal{G}$ of $G = S \times R$. Let

$$\mathcal{S} = \mathcal{J} + \mathcal{K} + \mathcal{L}$$

be the decomposition of $\mathcal{S}$ as discussed in Section 3.1 (with the difference that $\mathcal{H}$ is now $\mathcal{J}$, $\mathcal{G}$ is now $\mathcal{S}$, and $\mathcal{S}$ is now $\mathcal{L}$)

$$G = S + R = J + K + L + R.$$

Now, recall that $H = \{(s, \lambda(s)) : s \in I\}$ and therefore

$$H = \{v + \lambda'(v) : v \in J\},$$

where λ' is the derived Lie algebra homomorphism. It follows that

$$G = H + K + L + R$$

and one can easily check that this is a reductive decomposition. Our scalar field determines thus an Ad (H)-invariant scalar product on $K + L + R$. The scalar product on $K + L$ clearly comes from g_E and the one on R from δ_R. It follows that ω_U determines scalar product (K, R) and (L, R) or, equivalently, linear maps $K \to R$ and $L \to R$ where we have identified R with its dual using δ_R.

6.2. Geometry of scalar fields

Let us take a closer look into the geometry of scalar fields. Here again we find a place where the Reduction Theorem finds its natural application. First of all, we know that the scalar fields describe a G-invariant metric on $H\backslash G$. But now $G = S \times R$ and $H = \{(s, \lambda(s)) : s \in I\}$. Let us see that now *$H\backslash G$ is a principal bundle over $I\backslash S$ with structure group R*. The projection $\pi : H\backslash G \to I\backslash S$ is defined by $\pi : [(s, r)] \to [s]$, where $s \in S$, $r \in R$, and the brackets [ ] stand for H and I cosets respectively. Action of the structure group R is given by $[(s, r)]r' \equiv [(s, rr')]$. Finally S acts on the bundle by bundle automorphisms $t[(s, r)] \equiv [st^{-1}, r)]$ which induce the canonical action of S on the base $I\backslash S$. Our scalar field is a metric on the bundle which is, in particular, R-invariant. Thus, we know, it determines: metric on the base $I\backslash S$, connection on $I\backslash S$ with gauge group R, and R-invariant metric on R. Let us describe this explicitly. We use indices α, β for $K + L \simeq S/J$, and indices i, j for R. Then $g_{H\backslash G}$ decomposes into

$$g_{\alpha\beta} = h_{\alpha\beta} + \phi^i_\alpha \phi^j_\beta \delta_{ij}, \qquad g^{\alpha\beta} = h^{\alpha\beta}$$

$$g_{\alpha j} = \psi^i_\alpha \delta_{ij}, \qquad g^{\alpha i} = + h^{\alpha\beta}\phi^i_\beta$$

$$g_{ij} = \delta_{ij}, \qquad g^{ij} = \delta^{ij} + \phi^i_\alpha \phi^j_\beta h^{\alpha\beta},$$

where δ_{ij} are the components of the Killing metric on R, $h_{\alpha\beta}$ is the induced (still S-invariant) metric on $I\backslash S$, and - ϕ^i_α is the gauge field on $I\backslash S$ with gauge group R. It is the field ϕ^i_α which comes from the reduction of the original Yang--Mills field ω_U. It satisfies the constraint of Ad (H)-invariance

$$\phi \circ \mathrm{Ad}\,(s) = \mathrm{Ad}\,(\lambda(s)) \circ \phi,$$

or, infinitesimally,

$$C^{\beta}_{\underline{\alpha}\alpha}\phi_{\beta} = \lambda^{i}_{\underline{\alpha}}\phi^{j}C^{k}_{ij},$$

where the index $\underline{\alpha}$ runs through J. Since ϕ intertwines the representations of I on $K + L$ and on R (via λ), and since the representation of I on K is trivial, it follows that $\phi(K)$ must commute with $\lambda(I)$ i.e. $\phi(K) \subset Z \equiv \mathrm{Lie}\,(Z)$. Manton used tha scalar field ϕ^{i}_{α} to describe Higgs fields of the Weinberg-Salam model [29]. In his example K is trivial.

6.3. Results

We end this section with giving the result of the dimensional reduction of the scalar curvature R_U with the Killing metric for R. For details see [30]. We have

$$R_U = R_E + YM_{E,R} + R_R = $$
$$= R_M + YM_{M,N(H)|H} + L(h_{\dot{\alpha}\beta}) + L(\phi),$$

with

R_E – scalar curvature of E
$YM_{E,R}$ – Yang-Mills Lagrangian for (U, π, E, R)
R_R – scalar curvature of R endowed with the Killing metric (constant, see (3.6.d))
R_M – scalar curvature of the metric $g_{\mu\nu}$ on M
$YM_{M,N(H)|H}$ – Yang-Mills Lagrangian for $(Q, \pi, M, N(H)\,|\,H)$
$L(h_{\alpha\beta})$ – Lagrangian for scalar fields from g_E (see the second, fourth and fifth terms of (4.4.4) replacing $g_{\alpha\beta}$ by $h_{\alpha\beta}$).

The term $L(\phi)$ containing the fields ϕ^{i}_{α} is

$$L(\phi) = \mathrm{Kin}\,(\phi) + V(\phi) + \delta(\phi), \quad \text{where}$$

$$\mathrm{Kin}\,(\phi) = -\,1/2\, h^{\alpha\alpha'}\delta_{ii'}D_{\mu}\phi^{i}_{\alpha}D^{\mu}\phi^{i'}_{\alpha'},$$

$$V(\phi) = -\frac{1}{4}\, h^{\alpha\alpha'}h^{\beta\beta'}\delta_{ii'}(C^{\gamma}_{\alpha\beta}\phi^{i}_{\gamma} + C^{\gamma}_{\alpha\beta}\lambda^{i}_{\gamma} - C^{i}_{jk}\phi^{j}_{\alpha}\phi^{k}_{\beta}) \times$$
$$\times (C^{\gamma'}_{\alpha'\beta'}\phi^{i'}_{\gamma'} + C^{\gamma'}_{\alpha'\beta'}\lambda^{i'}_{\gamma'} - C^{i'}_{j'k'}\phi^{j'}_{\alpha'}\phi^{k'}_{\beta'}),$$

$$\delta(\phi) = -\frac{1}{4}\,\delta_{\hat{i}\hat{j}}\phi^{\hat{i}}_{\hat{\alpha}}F^{\mu\nu,\hat{\alpha}}(\phi^{\hat{j}}_{\hat{\beta}}F^{\hat{\beta}}_{\mu\nu} - 2F^{\hat{j}}_{\mu\nu}),$$

where $\hat{i}, \hat{j}$ run through K and i, j through L. Thus, when K is nontrivial we have a *nonminimal* interaction term for $\phi : K \to Z$.

REMARK. It should be stressed again that the dimensional reduction of Einstein-

-Yang-Mills system by imposing symmetry group S is equivalent to the dimensional reduction of a metric with *nonsimple* symmetry group and *diagonal* imbedding of the isotropy group.

7. COMMENTS ON RELATED ROPICS

7.1. Einstein-Cartan theory and spinors

To discuss spinors we must go beyond metric formalism and introduce the vielbein bundle. There are subtle differences between different approaches to this problem. We choose one which allows us to teat spinors *even if space-time does not admit a spinor structure*.

Let E be an n-dimensional manifold, let $\eta = \mathrm{diag}\,(+1, +1, \ldots, -1, -1, \ldots)$ be the standard flat-space metric, and let $\lambda : \mathrm{Spin}\,(\eta) \to SO(\eta)$ the twofold spin covering. Let $(P, \pi, E, \mathrm{Spin}\,(\eta))$ be a principal bundle over E with structure group $\mathrm{Spin}\,(\eta)$. (Such a bundle always exists, for example we can take the trivial bundle $E \times \mathrm{Spin}\,(\eta)$). Let $\rho : \mathrm{Spin}\,(\eta) \to \mathrm{End}\,(V)$ be a faithfull representation of $\mathrm{Spin}\,(\eta)$ in (real or complex) vector space V. The associated bundle $P \times_\rho V$ is called the bundle of spinors of type ρ. The group homomorphism λ may be considered as a (non-faithful) representation of $\mathrm{Spin}\,(\eta)$ in $\mathbb{R}^n$. We can therefore build the associated bundle $P \times_\lambda \mathbb{R}^n$. The dynamical variables of the Einstein-Cartan theory are

i) principal connection ω in P

ii) one-form $\Theta : TE \to P \times_\lambda \mathbb{R}^n$ on E with values in the bundle $P \times_\lambda \mathbb{R}^n$.

We can also add spinorial matter as

iii) cross-section ψ of the vector bundle $P \times_\rho V$ of spinors.

How do we build a Lagrangian for this theory? Θ is a one-form on E with values in the associated bundle. The curvature $\Omega = D\omega$ can be considered as two-form on E with values in $P \times_{\mathrm{Ad}} so(\eta)$ corresponding to the adjoint representation of $\mathrm{Spin}\,(\eta)$ on its Lie algebra $\mathrm{spin}\,(\eta) = so(\eta)$. The Lagrangian n-form is then given by

$$L_{EC} = \epsilon_{a_1 \cdots a_{n-2} a_{n-1} a_n} \Theta^{a_1} \wedge \ldots \wedge \Theta^{a_{n-2}} \wedge \Omega^{a_{n-1} a_n},$$

where $\Omega^{ab} = \Omega^a{}_c \eta^{cb}$. Of course L_{EC} was written with respect to a local cross--section of P - Θ has now indices and Ω also - but the Lagrangian is clearly independent of σ. This because ϵ and η are invariant tensors of $SO(\eta)$.

It is crucial that *no catastrophy occurs* even if $\det \Theta^a_\mu(x) = 0$ at some point. Θ is called a soldering form and sometimes vielbein. In our formulation vielbein may be allowed to become degenerate and even vanish. We can say even more: Θ *must* become degenerate at least at one point if the topology of E does not allow for a spinor structure. What is more interesting is that *no catastrophy occurs also to* *the Dirac Lagrangian when vielbein degenerates or even vanishes*. The Dirac Lagrangian is described as follows. We assume that in V we have given not only a representation ρ of Spin (η) but that ρ is derived from a representation of the Clifford algebra $C(\eta)$. If γ_a are the endomorphisms of V representing the basis vectors of $\mathbb{R}^n$, then

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}$$

and

$$\rho(r)\,\gamma_a \rho(r)^{-1} = \gamma_b \lambda(r)^b{}_a, \qquad r \in \mathrm{Spin}\,(\eta).$$

We also assume that in V we have given a bilinear (or sesquilinear, if V is complex) scalar product invariant under ρ (Spin (η)). Then

$$L_{\mathrm{Dirac}} = \epsilon_{a_1 \ldots a_n} \Theta^{a_1} \wedge \ldots \wedge \Theta^{a_{n-1}} \wedge \tau^{a_n}$$

with

$$\tau^{a_n} = (\psi, \gamma^{a_n} D\psi),$$

where $D\psi$ is the covariant derivative of ψ with respect to the principal connection ω. It is evident that Θ may become degenerate without doing any harm also to this Lagrangian. I hope to discuss these problems elsewhere. Here let us consider the problem of spinor fields with symmetries.

7.2. S-invariant spinors

Since spinors live in a bundle associated to P, their symmetries must be described in terms of vector fields on P and not on E. Thus we meet again (see Section 5) the problem of lifting symmetries from the base to the bundle space. Here however this problem is easily solved if Θ is everywhere of the maximal rank. Indeed, it is easy to see that then *for every vector field X on E there exists a unique invariant lift $\widetilde{X}$ of X to P such that* $L_{\widetilde{X}}\Theta = 0$. Now, a spinor field ψ can be also considered as an equivariant function on P with values in V:

$$\psi(pr) = \rho(r^{-1})\,\psi(p), \qquad r \in \mathrm{Spin}\,(\eta).$$

Therefore we can call X a symmetry of ψ if

$$L_{\widetilde{X}}\,\psi = 0,$$

where $\widetilde{X}$ is determined by $L_{\widetilde{X}}\Theta = 0$, $\pi_* \widetilde{X} = X$, and the invariance of $\widetilde{X}$. In general however, when det (Θ) is allowed to vanish we can not avoid the lifting problem and we have to assume that action of the symmetry group S on the bundle P by automorphisms is somehow given. We can write an integral version of the last formula

$$\psi(sp) = \psi(p), \qquad s \in S.$$

Invariant spinors on E can then be described in terms of spinor multiplets on $M \equiv E/S$. However, we can easily relax the last condition without loosing its nice properties. Assume in V we have also a representation α of the group S *which commutes* with ρ (we have not assumed that ρ is irreducible). Then ψ is called *α-equivariant* if

$$\psi(sp) = \alpha(s)\psi(p).$$

For example, in five-dimensional Kaluza-Klein theory S may be taken $U(1)$, and $\alpha = \alpha_n$ a character of $U(1)$ i.e. $\alpha_n(\phi) = \exp(in\,\phi)$. Then α_n-equivariant spinor on E is a spinor on M carrying electric charge ne. (See Ref. [49]).

7.3. Color and Higgs charges

In a simple Kaluza-Klein theory on a principal bundle, and without scalar fields, it is well known that geodesics in the bundle project onto the trajectories in M which describe particles with a color charge interacting with non-Abelian Yang-Mills field via Lorentz-type force (see e.g. [31], [32]). Consider now a more general case when G acts on E with orbits of type $H\backslash G$ and scalar fields $g_{\alpha\beta}$. Assuming metric in E to be G-invariant we know that the effective gauge group is $N(H)|H$ and $g_{\alpha\beta}(x)$ describe G-invariant metric in $H\backslash G$ - the shape of the copy of $H\backslash G$ at x. Let us also recall that the field $g_{\alpha\beta}$ splits into two kinds: $g_{\alpha\beta} = (g_{\hat{a}\hat{b}}, g_{ab})$, where $(g_{\hat{a}\hat{b}})$ is a scalar product on K, and (g_{ab}) an $\mathrm{Ad}(H)$-invariant scalar product on L. Here K is the space of all $\mathrm{Ad}(H)$ singlets in G/H and L is the $\mathrm{Ad}(H)$-invariant complement of K in G/H (see Section 4.1). The Christoffel symbols of the Levi-Civita connection have been already calculated so that it is easy to write down geodesic equations in E. A careful discussion is however necessary to analyze their projections on M. Let us give here the results (for details see [33], [34]). The projected trajectory describes a particle with two charges: a color charge $q^{\hat{a}}$ and Higgs charge λ^a. Both take values in associated bundles; q in $P \times_{\mathrm{Ad}} K$ and λ in $P \times_{\mathrm{Ad}} (L/H)$. The equations of motion are

$$\frac{D\dot{x}^\mu}{\mathrm{d}t} = q_{\hat{a}} F^{\hat{a}}_{\mu\nu} x^\nu + \frac{1}{2}\, q^{\hat{a}} q^{\hat{b}} D_\mu(g_{\hat{a}\hat{b}}) + \frac{1}{2}\, \lambda^a \lambda^b D_\mu g_{ab}$$

Non-Abelian	Type I	Type II
Lorentz force	Higgs force	Higgs force

$$\frac{Dq_{\hat{a}}}{\mathrm{d}t} = C_{\hat{a}\hat{b},\hat{c}}\, q^{\hat{b}} q^{\hat{c}} + C_{\hat{a}b,c}\, \lambda^b \lambda^c$$

Type I	Type II
charge nonconservation	charge nonconservation

$$\frac{D\lambda_a}{\mathrm{d}t} = C_{ab,c}\lambda^b \lambda^c .$$

It is interesting that the Higgs charge λ has a geometrical interpretation of describing the slope of the particle trajectory with respect to the principal bundle P imbedded in E. Observe however that the word «charge» really means «charge/mass ratio» here.

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