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CONSISTENCY OF THE G-INVARIANT KALUZA-KLEIN SCHEME

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We consider the G-invariant Kaluza-Klein scheme on $M \times G/H$ leading to the $N(H)/H$ gauge group and demonstrate its consistency. The full-scale ansatz with $G \times N(H)/H$ gauge bosons from $M \times G/H$ compactification is argued to be, in general, inconsistent.

1. Introduction

The present paper is a continuation and, in a sense, also a closure of a series of papers [1–5] in which we investigated the geometrical meaning of "dimensional reduction" – a procedure for obtaining an effective four-dimensional multifield theory from a multidimensional uni- (or "few") field theory. In this series of papers, we have given geometrical foundations to a whole family of theories of the Kaluza-Klein type under the assumption that the internal spaces are orbits of a certain global isometry group G, thus being homogeneous spaces of the type G/H. In the simplest model of this type, one considers just one field in a multidimensional universe – the metric tensor, with lagrangian $(R - 2\Lambda)\sqrt{g}$. This simple Kaluza-Klein theory was investigated in detail in ref. [2], and it will be sufficient for our purposes to restrict our attention mainly to this model also in the present paper*. The message coming from the results of ref. [2] can be summarized as follows: consider spontaneous compactification on $M \times S$ where the ground state has (internal) symmetry group G' acting transitively on S, and let G be a subgroup of G' such

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* Adding primary Yang-Mills fields to this scheme can be achieved using the results of refs. [3] and [4], and the addition also of antisymmetric tensor fields and spinors (as is necessary in supersymmetric models) can be carried out using ref. [5].

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that $S = G/H$ (i.e. G is transitive on S). Then, taking into account all G-invariant modes, and only those, of the metric in the $(m+s)$-dimensional universe $E \simeq M \times (G/H)$, M being m-dimensional space-time, one obtains an effective m-dimensional theory containing a metric tensor of M, a Yang-Mills field with gauge group $N(H)/H$, $N(H)$ being the normalizer of H in G, and, also, a certain multiplet of scalar fields whose composition and colour content depend on G and on H. Now, in ref. [6] it has been shown, using general arguments, that a G-invariant ansatz should always be consistent provided that one takes into account all G-invariant modes. The discussion of the consistency problem given in refs. [6] and [7] (see also ref. [8] for a review of recent results) seems to indicate also that the most popular ansatz (see, for example, refs. [9–12]) involving Killing vectors, which is explicitly *not* G-invariant, should be considered as guilty of inconsistency unless proved innocent for a specific model*.

In the present paper, we tackle this consistency problem with the following results:

(i) We show by explicit calculations that dimensional reduction (from $M \times G/H$ to M), based on the G-invariant ansatz for the metric tensor [2], and thus giving rise to Yang-Mills fields with gauge group $N(H)/H$, leads indeed to the effective m-dimensional lagrangian and field equations which are fully consistent with the original $(m+s)$-dimensional $(s = \dim(G/H))$ theory.

(ii) On the other hand, following certain geometrical considerations of ref. [13], we consider another ansatz, also of a geometrical nature but more general than the first one, which gives rise to an effective m-dimensional theory with gauge group $N(H)/H \times G$ (i.e. the maximal one allowed by the geometry of G-action). Here we argue that the resulting m-dimensional theory is, in general, inconsistent with the original one.

We wish to close this introductory section with a few comments:

(1) A Kaluza-Klein scheme is called "consistent" if (a) it admits a compactifying ground state solution; (b) if every solution of the resulting m-dimensional field equations can also be interpreted as a solution of the original $(m+s)$-dimensional theory**.

(2) A scheme which is inconsistent need not necessarily be all wrong; however, such an inconsistency suggests that "something" goes wrong with the truncation scheme.

(3) A scheme which is consistent must still be shown to be energetically stable before being accepted as a candidate for a realistic theory. We do not mean that an inconsistent ansatz is physically meaningless and we do not intend to discuss here the possible physical relevance of such an ansatz.

* The several known cases of a consistent non-G-invariant ansatz are discussed in ref. [8].

** Notice that (a) is implied by (b); we are grateful to M. Duff for discussions of the consistency problem.

2. Explicit form of the G-invariant ansatz

We first warn the reader that in the following, all our considerations will be local; global properties, bundle structure and all that were already discussed in ref. [2]. Also, we want to make the form of our exposure correspond to its content; therefore we will use the most appropriate local "jargon". With that in mind, let us consider the Einstein-Hilbert lagrangian for g_{AB} ($A, B = 1, 2, \ldots, d = m + s$) with a cosmological constant $(R - 2\Lambda)\sqrt{g}$ in $d = m + s$ dimensions. We consider this model being fully aware of all its non-realistic features, and we consider it explicitly just for the reason that it is *the* simplest model (or rather class of models) which exhibits consistent dimensional reduction; at the same time, it suits us best as an illustration of methods and results of G-invariant dimensional reduction.

The model admits a spontaneous compactification on $\mathrm{E} = \mathrm{M} \times (\mathrm{G}/\mathrm{H})$ with M being, for instance, a de Sitter space and G/H a homogeneous Einstein space. Here G can be any compact subgroup of the isometry group of the vacuum (ground state metric), acting transitively on the internal space. To get an effective m-dimensional theory describing the dynamics of zero modes of g_{AB}, it was proposed in ref. [2] to take into account all G-invariant configurations $g_{AB}(x, y)$ on $\mathrm{E} = \mathrm{M} \times \mathrm{G}/\mathrm{H}$, and only those. Explicitly, this ansatz of ref. [2] can be described as follows:

$$\hat{g}_{\mu\nu}(x, y) = g_{\mu\nu}(x), \tag{2.1}$$

$$\hat{g}_{\mu\alpha}(x, y) = 0, \tag{2.2}$$

$$\hat{g}_{\alpha\beta}(x, y) = \Lambda^{\gamma}_{\alpha}(a)\Lambda^{\delta}_{\beta}(a)h_{\gamma\delta}(x). \tag{2.3}$$

The rest will consist exclusively of the necessary elucidations of the above formulae.

(i) The metric $\hat{g}_{AB}(x, y)$ is written in the basis consisting of vectors e_μ ($\mu = 1, \ldots, m$) and K_α ($\alpha = 1, \ldots, s = \dim(\mathrm{G}/\mathrm{H})$) where

$$e_\mu(x, y) = \frac{\partial}{\partial x^\mu} - A^\alpha_\mu(x, y)K_\alpha(y), \tag{2.4}$$

and $K_\alpha(y)$ is a basis of the Killing vectors of G/H. More precisely, let K_i ($i = 1, \ldots, n = \dim \mathrm{G}$) be the Killing vectors of the G-action on G/H [usually they are written as $K_i(y) = K^m_i(y)(\partial/\partial y^m)$, where y^m are coordinates on G/H]. These vector fields satisfy simple commutation relations

$$[K_i, K_j] = C^k_{ij}K_k, \tag{2.5}$$

but they form an overcomplete system on G/H, and one has to remove $(n - s)$ of them. To this end, one splits the Lie algebra of G into

$$\mathrm{Lie}(\mathrm{G}) = \mathrm{Lie}(\mathrm{H}) + \mathscr{S},$$

where $\mathscr{S}$ is a reductive complement (i.e. $h\mathscr{S}h^{-1}\subset\mathscr{S}$ for $h\in \mathrm{H}$) of Lie(H) in Lie(G) and, corresponding to this decomposition, one also splits a basis T_i in Lie(G) into $(T_{\underline{\alpha}}, T_\alpha)$, $T_{\underline{\alpha}}$ being a basis in Lie(H) and T_α a basis in $\mathscr{S}$. Then K_α – the Killing vectors corresponding to T_α – form a basis for vector fields on G/H in a neighbourhood of the origin of G/H.

(ii) The matrix $\Lambda^\alpha_\beta(a)$ entering the expression (2.3) is defined as follows. Let $\Lambda^i{}_j(a)$ be the matrix of the adjoint representation of G:

$$aT_ia^{-1}=\Lambda^j_i(a)T_j. \tag{2.6}$$

Then $\Lambda^\alpha_\beta(a)$ is the submatrix of $\Lambda^i{}_j$ corresponding to $i=\alpha,\ j=\beta$.

(iii) The matrix $h_{\alpha\beta}(x)$ of scalar fields which defines $g_{\alpha\beta}(x,y)$ is subjected to the linear constraint of Ad(H)-invariance:

$$\Lambda^\gamma_\alpha(a)\Lambda^\delta_\beta(a)h_{\gamma\delta}(x)=h_{\alpha\beta}(x) \tag{2.7}$$

for all $a\in \mathrm{H}$. Infinitesimally this is expressed by the vanishing of the following Lie derivatives:

$$\mathscr{L}_{\underline{\alpha}}h_{\beta\gamma}(x)\equiv C^\delta_{\underline{\alpha}\beta}h_{\delta\gamma}(x)+C^\delta_{\underline{\alpha}\gamma}h_{\beta\delta}(x)=0. \tag{2.8}$$

How to solve these constraints and how to count the independent degrees of freedom has been discussed in ref. [2].

(iv) The group element $a\in \mathrm{G}$ which appears in (2.3) is any representative of the coset $y=Ha$[★]. That the r.h.s. of (2.3) does not depend on the choice of such a representative is automatically guaranteed by the constraints (2.7).

(v) The quantity $A^\alpha_\mu(x,y)$ in (2.4) is defined by

$$A^\alpha_\mu(x,y)=\Lambda^\alpha_{\hat{a}}(a)A^{\hat{a}}_\mu(x), \tag{2.9}$$

where $A^{\hat{a}}_\mu(x)$ are the Yang-Mills potentials. The meaning of the index $\hat{a}$ is explained as follows:

The subspace $\mathscr{S}$ of Lie(G) is further decomposed into $\mathscr{S}=\mathscr{K}+\mathscr{L}$, where $\mathscr{K}$ consists of all Ad(H) singlets in $\mathscr{S}$, and $\mathscr{L}$ is a reductive complement of $\mathscr{K}$ in $\mathscr{S}$ (i.e. $[\mathscr{K},\mathscr{L}]\subset\mathscr{L}$ and $[\mathscr{K},\mathscr{L}]\subset\mathscr{L}$). The basis T_α in $\mathscr{S}$ is then further split into $T_{\hat{a}}\in\mathscr{K}$ and $T_a\in\mathscr{L}$. $\mathscr{K}$ is now a Lie subalgebra of Lie(G) – in fact it is the Lie algebra of the effective gauge group which is N(H)/H. The Yang-Mills potential A_μ has values in $\mathscr{K}$. We warn the reader that $\mathrm{N(H)}\doteq\{a\in \mathrm{G}: aHa^{-1}=H\}$ – the *normalizer* of H in G is not the same as the centralizer.

(vi) The basis (e_μ, K_α) which has been used for writing down the metric (2.1)–(2.3) is one which is most convenient for calculations. In this basis the vectors e_μ are

★ In this paper we use "G/H" to denote *right* coset space.

orthogonal to K_α, but this is not in general an orthonormal basis; neither e_μ nor K_α are assumed to be orthogonal between themselves. Usually one writes the Kaluza-Klein ansatz in the basis (∂_μ, K_α) or even $(\partial_\mu, \partial_m)$. The change of basis is easily accomplished with the formula (2.4). Thus, for example, in the basis (∂_μ, K_α), the r.h.s. of (2.1) would be

$$g_{\mu\nu}(x) + h_{\hat{a}\hat{b}}(x) A^{\hat{a}}_\mu(x) A^{\hat{b}}_\nu(x),$$

while that of (2.2) would read

$$\Lambda^{\hat{a}}_\alpha(a) h_{\hat{a}\hat{b}}(x) A^{\hat{b}}_\mu(x).$$

(vii) The price paid for the simple form of (2.1)–(2.3) is in the more complicated form of the commutators. In order to do any calculations (in particular, to compute the Christoffel symbols and the curvature tensor), one needs the commutators of the basis. Here they are given by

$$[e_\mu, e_\nu](x, y) = -\Lambda^\alpha_{\hat{a}}(a) F^{\hat{a}}_{\mu\nu}(x) K_\alpha(y),$$

$$\left[e_\mu, K_\alpha\right] = 0,$$

$$\left[K_\alpha, K_\beta\right](y) = f^\gamma_{\alpha\beta}(a) K_\gamma(y),$$

where $F^{\hat{a}}_{\mu\nu}$ is the field strength of $A^{\hat{a}}_\mu$

$$F^{\hat{a}}_{\mu\nu} = \partial_\mu A^{\hat{a}}_\nu - \partial_\nu A^{\hat{a}}_\mu + C^{\hat{a}}_{\hat{b}\hat{c}} A^{\hat{b}}_\mu A^{\hat{c}}_\nu$$

and the structure functions $f^\gamma_{\alpha\beta}(a)$ are given by

$$f^\gamma_{\alpha\beta}(a) = C^\gamma_{\alpha\beta} + C^{\underline{\gamma}}_{\alpha\beta} \Lambda^\delta_{\underline{\gamma}}(a) L^\gamma_\delta(a),$$

where $L^\alpha_\beta(a)$ is the inverse matrix of $\Lambda^\alpha_\beta(a)$. Here, as everywhere in this paper, the relation between $y \in G/H$ and $a \in G$ is given by $y = y_0 a$, where $y_0 = H = [e]$ is the origin of G/H. Everywhere, care is taken so that the formulae we use do not depend on the choice of a.

3. Consistency of the G-invariant ansatz

The formulae of the previous section allow us to compute the Christoffel symbols, curvature, Ricci and Einstein tensors of $\hat{g}_{AB}(x, y)$ *for* y *in a neighbourhood of the origin of* G/H – too far from the origin, the K_α can fail to be linearly independent. But, in fact, it is enough to perform these calculations for $y = y_0$, y_0 being *the* origin of G/H. This is due to the G-invariance of the metric which implies

$$\hat{g}_{AB}(x, y) = \Lambda^C_A(a^{-1})\Lambda^D_B(a^{-1})\hat{g}_{CD}(x, y_0), \tag{3.1}$$

where $y = y_0 a$, and $\Lambda^A_B(a) \doteq (\delta^\mu_\nu, \Lambda^\alpha_\beta(a))$. Analogous formulae hold also for the curvature, Ricci and Einstein tensors. In particular, the scalar curvature R is constant along y and depends on x only. The results of the calculation of $\hat{R}_{AB}(x, y_0)$ and $\hat{R}(x)$ read*

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}F^{\hat{a}}_{\mu\sigma}F^{\hat{a}}_{\nu\sigma} - \tfrac{1}{2}h^{\alpha\beta}h^{\gamma\delta}D_\mu h_{\alpha\gamma}D_\nu h_{\beta\delta} - \tfrac{1}{2}D_\mu h_\nu, \tag{3.2}$$

$$\hat{R}_{\mu\alpha} = \tfrac{1}{2}D_\sigma F_{\mu\sigma,\alpha} + \tfrac{1}{4}h_\sigma F_{\mu\sigma,\alpha} - \tfrac{1}{2}C^\gamma_{\alpha\beta}h^{\beta\delta}D_\mu h_{\gamma\delta}, \tag{3.3}$$

$$\begin{aligned}\hat{R}_{\alpha\beta} = {} & R_{\alpha\beta}(\mathrm{G/H}) + \tfrac{1}{4}F_{\mu\nu,\alpha}F_{\mu\nu,\beta} + \tfrac{1}{2}h^{\gamma\delta}D_\mu h_{\alpha\gamma}D_\mu h_{\beta\delta} \\ & - \tfrac{1}{4}h_\mu D_\mu h_{\alpha\beta} - \tfrac{1}{2}D_\mu D_\mu h_{\alpha\beta},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\hat{R} = {} & R + R(\mathrm{G/H}) - \tfrac{1}{4}F_{\mu\nu,\hat{a}}F_{\mu\nu,\hat{a}} - \tfrac{1}{4}h^{\alpha\beta}h^{\gamma\delta}D_\mu h_{\alpha\gamma}D_\mu h_{\beta\delta} \\ & - \tfrac{1}{4}h_\mu h_\mu - D_\mu h_\mu,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}R_{\alpha\beta}(\mathrm{G/H}) = {} & \tfrac{1}{4}C_{\gamma\delta,\alpha}C_{\gamma\delta,\beta} - \tfrac{1}{2}C_{\alpha\gamma,\delta}C_{\beta\gamma,\delta} - \tfrac{1}{2}C_{\alpha\gamma,\delta}C_{\beta\delta,\gamma} \\ & - \tfrac{1}{2}C^\delta_{\alpha\gamma}C^\gamma_{\beta\underline{\delta}} - \tfrac{1}{2}C^\delta_{\beta\gamma}C^\gamma_{\alpha\underline{\delta}} - \tfrac{1}{2}(C_{\gamma\alpha,\beta} + C_{\gamma\beta,\alpha})C^\delta_{\gamma\delta},\end{aligned} \tag{3.6}$$

$$\begin{aligned}R(\mathrm{G/H}) = h^{\alpha\beta}R_{\alpha\beta}(\mathrm{G/H}) = {} & -\tfrac{1}{4}C_{\alpha\beta,\gamma}C_{\alpha\beta,\gamma} - \tfrac{1}{2}C_{\alpha\beta,\gamma}C_{\alpha\gamma,\beta} \\ & - C^\delta_{\beta\alpha}C^\alpha_{\beta\underline{\delta}} - C^\alpha_{\delta\alpha}C^\beta_{\delta\beta}.\end{aligned} \tag{3.7}$$

The summations over the repeated indices on the same level, as well as all raisings and lowerings of indices are always carried out using $g_{\mu\nu}$ and $h_{\alpha\beta}$ or their inverses, for example $C_{\alpha\beta,\gamma} = h_{\gamma\gamma'}C^{\gamma'}_{\alpha\beta}$. The D_μ's are the covariant derivatives employing Yang-Mills potentials and, when necessary, also the Christoffel symbols of M (as in the first term of (3.3) and the last term of (3.4)); so, for example,

$$D_\mu h_{\alpha\beta} \doteq \partial_\mu h_{\alpha\beta} - A^{\hat{a}}_\mu(C^\gamma_{\hat{a}\alpha}h_{\gamma\beta} + C^\gamma_{\hat{a}\beta}h_{\alpha\gamma}).$$

We also used the notation h_μ to denote $h^{\alpha\beta}D_\mu h_{\alpha\beta}$. Finally, the $F^\alpha_{\mu\nu}$ entering (3.3) and (3.4) are non-zero only for $\alpha = \hat{a}$. From now on, we will assume that the group G is unimodular, then the last term of (3.6) and (3.7) vanishes ($C^\delta_{\gamma\delta} = 0$).

* Everywhere in this paper we follow the conventions of ref. [14].

The field equations resulting from the lagrangian $(\hat{R}-2\Lambda)\sqrt{g}$ are

$$\hat{E}_{AB}+\Lambda\hat{g}_{AB}=\hat{T}_{AB},$$

where $\hat{E}_{AB}\doteq\hat{R}_{AB}-\frac{1}{2}\hat{R}\hat{g}_{AB}$, and $\hat{T}_{AB}$ describes the contribution of matter fields. In the simple model we consider $T_{AB}\equiv 0$, and so the field equations become

$$\hat{R}_{\mu\nu}=+\frac{2\Lambda}{d-2}\hat{g}_{\mu\nu}, \tag{3.8i}$$

$$\hat{R}_{\mu\alpha}=0, \tag{3.8ii}$$

$$\hat{R}_{\alpha\beta}=+\frac{2\Lambda}{d-2}\hat{g}_{\alpha\beta}, \tag{3.8iii}$$

i.e. they describe an Einstein space with Einstein constant $2\Lambda/(d-2)$, $d=m+s$. Because of the aforementioned simple "propagation law" (3.1), it is enough to satisfy the field equations (3.8) only at $y=y_0$, where the l.h.s. is given by (3.2)–(3.4) and the r.h.s. by (2.1)–(2.3), with $y=y_0$ and $a=e$.

To check the consistency of the ansatz, we have to compare the equations (3.8), which are now equations for $g_{\mu\nu}(x)$, $A_\mu^{\hat{a}}(x)$ and $h_{\alpha\beta}(x)$, with the ones obtained from the m-dimensional actions which results from averaging the original lagrangian over the internal coordinates. Modulo a constant proportionality factor (related to the standard volume of G/H), this m-dimensional action is

$$A_{\text{aver}}=\int(\hat{R}-2\Lambda)g^{1/2}h^{1/2}\,\mathrm{d}^4x, \tag{3.9}$$

where g and h denote the determinants of $g_{\mu\nu}$ and $h_{\alpha\beta}$ respectively. This action should now be varied with respect to $g_{\mu\nu}(x)$, $A_\mu^{\hat{a}}(x)$ and $h_{\alpha\beta}(x)$ in order to obtain a set of m-dimensional equations for these fields. Here it is essential to take into account the constraints (2.8) by adding to (3.9) the Lagrange multiplier term $\int\lambda^{\underline{\alpha}}_{\beta\gamma}\mathscr{L}_{\underline{\alpha}}h^{\beta\gamma}\,\mathrm{d}^4x$. Using (3.5) and (2.8), one gets, by an explicit calculation, the following set of field equations:

$$\hat{R}_{\mu\nu}-\tfrac{1}{2}(\hat{R}-2\Lambda)g_{\mu\nu}=0, \tag{3.10}$$

$$\hat{R}_{\mu\hat{a}}=0, \tag{3.11}$$

$$g^{1/2}h^{1/2}\left[\hat{R}_{\alpha\beta}-\tfrac{1}{2}(\hat{R}-2\Lambda)h_{\alpha\beta}+\tfrac{1}{2}\left(C^{\underline{\delta}}_{\alpha\gamma}C^{\gamma}_{\beta\underline{\delta}}+C^{\underline{\delta}}_{\beta\gamma}C^{\gamma}_{\alpha\underline{\delta}}\right)\right]+\lambda^{\underline{\delta}}_{\alpha\gamma}C^{\gamma}_{\beta\underline{\delta}}+\lambda^{\underline{\delta}}_{\beta\gamma}C^{\gamma}_{\alpha\underline{\delta}}=0, \tag{3.12}$$

$$\mathscr{L}_{\underline{\alpha}}h^{\beta\gamma}=0. \tag{3.13}$$

While eq. (3.10) are evidently the same as (3.8i), eqs. (3.11) and (3.12) need further discussion. Consider first eq. (3.12). It is seen that by choosing the Lagrange multipliers to be

$$\lambda^{\delta}_{\alpha\gamma} = -\tfrac{1}{2} g^{1/2} h^{1/2} C^{\delta}_{\alpha\gamma}\,. \tag{3.14}$$

Eq. (3.12) becomes (3.8iii). Now, one knows that the Lagrange multipliers are uniquely determined by the constraints. On the other hand, the constraints (2.7) are compatible with (3.8iii); indeed, it is easy to see that $\hat{R}_{\alpha\beta}$ as given by (3.4) satisfies (2.7) if $h_{\alpha\beta}$ does. This justifies (3.14) and proves that (3.12) + (3.13) is the same as (3.8). Eq. (3.11) remains to be considered. It has exactly the form of (3.8ii), except that (3.8ii) asserts that $R_{\mu\alpha} = 0$ for all α, while (3.11) gives this conclusion only for $\alpha = \hat{a}$ (we remind the reader that the index α runs over all basis vectors of G/H while $\hat{a}$ runs only over those which span the Lie algebra of N(H)/H). Since $F_{\mu\nu,\alpha} \equiv 0$ for $\alpha \neq \hat{a}$, it remains to show that $X_{\mu\alpha} \doteq C^{\gamma}_{\alpha\beta} h^{\beta\delta} D_{\mu} h_{\gamma\delta} = 0$ for $\alpha \neq \hat{a}$. One can prove this as follows. First, using the Jacobi identity and the constraint equations (2.8), one can show that X^{α}_{μ}, with μ fixed, is a vector which is Ad(H)-invariant, and then use the fact that all Ad(H) singlets of $\mathscr{S}$ are in $\mathscr{K}$.

This ends the proof of the consistency of the G-invariant Kaluza-Klein scheme. Observe that what is essential in this statement is that every solution which is an extremum of the effective m-dimensional action is an extremum of the original d-dimensional action. The inverse statement, that is, that every *constrained* solution of the original field equations is a solution of the m-dimensional ones, follows directly from the fact that the effective m-dimensional action is defined as an integral of the original one over the internal variables.

Remark. One should not interpret $g_{\mu\nu}(x)$ (3.8) as an m-dimensional gravitational field. As is well known, one has first to perform a conformal rescaling of the fields $g_{\mu\nu}(x)$ and $h_{\alpha\beta}(x)$ by introducing new fields, $\bar{g}_{\mu\nu} = h^{r} g_{\mu\nu}$, $\bar{h}_{\alpha\beta} = h^{r} h_{\alpha\beta}$, where $h = \det h_{\alpha\beta}$ and $r = 1/(m-2)$. In terms of these new fields, the effective action becomes

$$A_{\text{aver}}\left[\bar{g}_{\mu\nu}, A^{\hat{a}}_{\mu}, \bar{h}_{\alpha\beta}\right] = \int \bar{g}^{1/2} \mathscr{L}[\bar{g}, A, \bar{h}],$$

where $\mathscr{L}[\bar{g}, A, \bar{h}]$ is now:

$$\mathscr{L}[\bar{g}, A, \bar{h}] = R + R(\mathrm{G/H}) - \tfrac{1}{4} F_{\mu\nu,\hat{a}} F_{\mu\nu,\hat{a}} - \tfrac{1}{4} \bar{h}^{\alpha\beta} \bar{h}^{\gamma\delta} D_{\mu} \bar{h}_{\alpha\beta} D_{\mu} \bar{h}_{\beta\delta}$$
$$+ \frac{1}{4(d-2)} \bar{h}_{\mu} \bar{h}_{\mu} - \frac{1}{\bar{g}^{1/2}} \partial_{\mu}\left(\bar{h}^{\mu} (\bar{h})^{-1/(d-2)} \bar{g}^{1/2}\right),$$

with $d = \dim(\mathrm{M} \times \mathrm{G/H}) = m + s$. Notice that the last term becomes an ordinary total derivative upon multiplication by $\bar{g}^{1/2}$.

4. Inconsistency of the "full-scale" ansatz

We have proved consistency of the G-invariant ansatz on $E = M \times (G/H)$. The effective m-dimensional theory which is consistent with the original theory in $d = m + s$ dimensions ($s = \dim(G/H)$) incorporates gauge fields with the group $N(H)/H$. Now, there is a widespread belief that the effective gauge group from G/H compactification should be G. It is our wish to show two things in this section:

(i) that there really exists an "ansatz" – we shall call it a "full-scale ansatz" – which (a) has a well-defined geometrical meaning, (b) gives the effective gauge group $N(H)/H \times G$;

(ii) that this "full-scale" ansatz is, unfortunately, in general, inconsistent.

The second conclusion applies in particular to the case of $E = M \times G$. The full-scale ansatz now predicts the gauge group $G_L \times G_R$ with $G_L = G_R = G$. G_L corresponds here to $N(H)/H$, while G_R corresponds to G. This ansatz is, in general, inconsistent; the source of inconsistency is the part of the metric which corresponds to the most popular (non-G-invariant) ansatz (see refs. [9–12]) and which gives rise to the gauge fields of G_R.

A very general geometrical construction giving the full-scale ansatz has been given elsewhere [13]. This construction is applicable to all cases where internal spaces have transitive isometry groups. Here, for reasons of simplicity, we will restrict our discussion to the case of $P = M \times G$ being a principal bundle. The following recipe then gives the full-scale ansatz:

(1) Artificially enlarge P to $\bar{P} = P \times G$.

(2) Now $G \times G$ acts on $\bar{P}$ by

$$(x, a, b)(c, d) = (x, ac, d^{-1}b). \tag{4.1}$$

(3) In particular, $G^{\text{diag}} \subset G \times G$ acts on $\bar{P}$ by

$$(x, a, b)(c, c) = (x, ac, c^{-1}b). \tag{4.2}$$

(4) Therefore $\bar{P}/G^{\text{diag}}$ is isomorphic to P; indeed, this isomorphism is given by $(x, a, b) \to (x, ab)$.

(5) Consider all metrics on P which are Kaluza-Klein projections from $\bar{P} = P \times G$ to P (in the sense discussed in sect. 2) of $G \times G$ invariant metrics on $\bar{P}$ (taken as $M \times (G \times G)$).

Notice that the remaining G action on P defined by $(x, a)(b) = (x, ab)$ comes from the following G action on $\bar{P} = M \times G \times G$:

$$(x, a, b)(c) = (x, a, bc). \tag{4.3}$$

This last action is not killed during the projection $\bar{P} \to P \simeq \bar{P}/G^{\text{diag}}$, whereas the action of $G \times G$ on $\bar{P}$ introduced in (4.1) no longer exists at the level of P. $G \times G$-invariant metrics on $\bar{P}$ (for the action (4.1)) are in particular G^{diag}-invariant and therefore do project on P on metrics which have no remaining invariance in general. Those very particular metrics of $\bar{P}$ which are $(G \times G) \times G$-invariant (for the actions (4.1) and (4.3)) or, at least, $G^{\text{diag}} \times G$-invariant, project onto G-invariant metrics on P. From the point of view of set theory, we have taken the product by G and then taken the quotient by G – we are again on P. But from the point of view of the field content, we have produced a large class of metrics on P which contains all G-invariant metrics, but also contains much more. This is the full-scale ansatz. It gives rise to gauge fields of $G_L \times G_R$, as mentioned above. It should be noticed that the scalar curvature of a given $G \times G$-invariant metric on $\bar{P}$ (for the action (4.1)) is only a function of $x \in M$ and can be "dimensionally reduced" directly to M by using the general technique of sect. 2 (the so-called G-invariant ansatz). However, the scalar curvature of the projected metric on P has no reason to be independent of $a \in G$, since it will not be G-invariant in general. One can now perform calculations of the scalar curvature associated with this more general, non-G-invariant ansatz. Hence, instead of (2.4), one has

$$e_\mu(x, y) = \frac{\partial}{\partial x^\mu} - A_\mu^\alpha(x, y) e_\alpha(y) - B_\mu^\alpha(x) e_\alpha(y). \tag{4.4}$$

We denote by $F_{\mu\nu}^\alpha$ and $G_{\mu\nu}^\beta$ the field strengths of gauge fields A_μ and B_μ respectively. This time we also have at our disposal not $(\frac{1}{2} \cdot n(n+1))$ but $(\frac{1}{2} \cdot (2n(2n+1))$ scalar fields ($n = \dim G$). The piece of scalar curvature which interests us now has the form

$$\hat{R} = R(M) - \tfrac{1}{4} g_{\alpha\beta}(x, a)\big(F_{\mu\nu}^\alpha(x, a) + G_{\mu\nu}^\alpha\big)\big(F_{\mu\nu}^\beta(x, a) + G_{\mu\nu}^\beta\big) + \cdots, \tag{4.5}$$

where $F_{\mu\nu}^\alpha(x, a)$ depends on a as in, for example, (2.9), and $g_{\alpha\beta}(x, a)$ is a function of scalar fields and a. For consistency, the solutions of equations of motion obtained from the lagrangian

$$\mathscr{L}\big[A_\mu, B_\mu, \gamma^{\mu\nu}, h_{\alpha\beta}, \dots\big] = \int_G \big[\hat{R}(x, a) - 2\Lambda\big] \mathrm{dvol}_G$$

should also be solutions of the set of equations (in P), $\hat{R}_{MN} = (2\Lambda/(d-2)) g_{MN}$. This last set of equations is a-dependent in general, whereas the first is not. With a special choice of scalar fields, one can make the dependence of $g_{\alpha\beta}$ on a reasonable:

$$g_{\alpha\beta}(x, a) = \Lambda_\alpha^\gamma(a) \Lambda_\beta^\delta(a) h_{\gamma\delta}(x).$$

But even with that choice, it is impossible to make the piece of R that contains the fields F and G to be a-independent *unless* $h_{\alpha\beta}(x) = \phi(x) k_{\alpha\beta}$, $k_{\alpha\beta}$ being the Killing metric of G [here we assume G simple and use the relation $\int \Lambda^{\beta}_{\alpha}(a)\Lambda^{\delta}_{\gamma}(a)\mathrm{dvol}_{G} = (\delta^{\beta\delta}\delta_{\alpha\gamma}/\dim G)\mathrm{vol}(G)$]. But this will be generally incompatible with the field equation containing $R_{\alpha\beta}$.

In the case where we start with a space $E = M \times G/H$, there exist (at least) two kinds of non-invariant Kaluza-Klein scenarios, both of them leading to an effective gauge theory incorporating a G-valued Yang-Mills field, both of them unfortunately inconsistent in general. These two scenarios have, however, a well-defined geometrical meaning that we sketch here.

First scenario. We enlarge artificially E into $\bar{E} = M \times G \times G/H$ (to make these considerations global, one should distinguish two subcases: if G acts globally on E, we build $\bar{E} = E \times G$, whereas if G does not act globally, we build $\bar{E} = P \times G/H$ where $P \simeq M \times G$ is the principal bundle associated to E). In either case, we have a well-defined $G \times G$ action on $\bar{E}$, the little group of which is $H \times \{e\}$; according to the general technique of sect. 2, we can build $G \times G$-invariant metrics on $\bar{E}$ and the effective Yang-Mills field emerging from this invariant ansatz will be $\bar{N}/\bar{H}$ where $\bar{N}$ is the normalizer of $\bar{H} = H \times \{e\}$ into $G \times G$, i.e. $\bar{N}/\bar{H} \simeq N/H \times G$. Equations obtained from direct dimensional reduction on M will, of course, be consistent with equations in $\bar{E}$. The above metrics on $\bar{E}$ are also G^{diag}-invariant and therefore go to the quotient $\bar{E}/G^{\mathrm{diag}} \simeq E$; the obtained metrics on E have usually no invariance left, and this describes the first kind of non-invariant ansatz on $M \times G/H$. Of course, there is no hope for this ansatz to be more "consistent" than the one already discussed in the special case where $H = \{e\}$.

Second scenario. We start with a G-invariant metric on $P = M \times G$ for the action $(x, a)b = (x, ab)$. Such a metric is a fortiori invariant under a subgroup H of G and goes to the quotient $E = P/H$. However, G invariance is lost after this operation and we run into the same kind of inconsistency problem as before. Notice that when P is not trivial (when it is not a product), G does not generally even act on E at all. The above two scenarios can be discussed globally (bundle structure, etc.), cf. refs. [13] and [15].

The attentive reader will have noticed that we did not attempt to prove that the most popular ansatz (the one leading to gauge group G when performing Kaluza-Klein dimensional reduction on G/H) is *always* inconsistent: miracles are possible depending on specific models (r.h.s. of field equations). However, as stated in the introduction, it should be considered as guilty of inconsistency, unless proved innocent for a specific model.

On the other hand, the G-invariant ansatz introduced in ref. [2] and leading to the effective gauge group N/H was shown to be consistent (in this paper we considered the simple case of a lagrangian $(R - 2\Lambda)\sqrt{g}$ in order for the ground space – an Einstein space – to be a solution of field equations, but the result is expected to be generally valid – compare the arguments given in ref. [6]).

The above facts seem to be now recognized under the name "consistent truncation in Kaluza-Klein theory" [6, 7].

Note added

In ref. [2], the factor $\frac{1}{2}$ in (3.5.7) should be $\frac{1}{4}$ (as in the CERN preprint). Also, the term "isotropy" on p. 97 six lines from the bottom should be replaced by ad H.

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