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Section A :
Physique théorique.

Conservation laws and string-like matter distributions (*)

by

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SUMMARY. — Equations of motion for singular distributions of matter, like point particles, strings, membranes and bags are derived by Souriau method. Interactions with metric tensor, non-Abelian gauge fields and G-structures are taken into account. Particles carrying spinorial charges in super-gravity field are also examined.

RÉSUMÉ. — On obtient par la méthode de Souriau des équations de mouvement pour des distributions singulières de matière, telles que des particules ponctuelles, des cordes, des membranes et des sacs. On incorpore des interactions avec le tenseur métrique, avec des champs de jauge non abéliens et des G-structures. On examine aussi le cas de particules portant des charges spinorielles dans un champ de supergravité.

1. INTRODUCTION

It is well known that the geodesic principle of general relativity can be derived from energy-momentum conservation, the latter being in turn

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a consequence of general covariance of the theory. Some authors (see e. g. [*1*] [*2*]) claim that also the equations of motion of the Nambu string can be derived by this method. We study singular distributions of matter, like point particles, strings, membranes, bags, etc. in a field of geometrical objects like metric tensor, gauge fields, G-structures. Instead of deriving relevant equations from conservation laws we follow a much simpler method of Souriau [*3*] which allows us to proceed directly from invariance to equations of motion. Nevertheless we have found it more convenient to change philosophy and appeal to Aristotel's Golden Rule of Mechanics rather than to « general covariance ». After formulation of a general framework in Sec. 2 we proceed to consider motions of charged singular distributions of matter in gravitational and non-Abelian gauge fields. For 1-dimensional distributions we get equations of Kerner and Wong [*4*] [*5*] [*6*] [*7*], and for 2-dimensional ones our equations contain those derived by Nielsen [*8*] from an action principle. In fact, as is discussed in Sec. 3 and 4, Nielsen's equations are stronger than ours since they specify internal energy-momentum tensor of the string in terms of its geometry and its current. Our analysis is to be compared with that given in [*1*] [*2*] where (apart of the fact that we include gauge fields not discussed there) the authors overlooked the fact that conservation laws do not determine string's dynamics unless its internal dynamics is specified so that Cauchy data's constraints become explicit and an appropriate phase space can be defined.

In Sec. 6 we discuss a wide class of theories where geometry is described in terms of a G-structure. It is found that a possibility of deriving a full dynamics from conservation laws, even for point particles, depends on the group G. Orthogonal groups are the best in this respect what distinguishes field theories based on multi-dimensional Riemannian geometries (endowed with any set of covariant constraints like e. g. Kaluza-Klein theories). Supergravity [*9*], considered as a constrained Lorentz structure on supermanifold, seems to have too poor a structure group to give deterministic equations of motion for a point particle endowed with mass and spinorial charge. Much better in this respect is metric supergravity [*10*] [*11*] although it may cause some other problems [*12*].

2. THE GOLDEN RULE

Let $\mathscr{R}$ be a space of geometries of certain kind and let $\mathscr{R}_\gamma$ be the tangent space to $\mathscr{R}$ at $\gamma \in \mathscr{R}$. Vectors $\delta\gamma \in \mathscr{R}_\gamma$ correspond to possible *displacements* of γ in $\mathscr{R}$, and linear forms on $\mathscr{R}_\gamma$ correspond to possible *matter distributions*. Usually each $\gamma \in \mathscr{R}$ is constrained to represent geometry of a fixed manifold $\mathscr{P}$ so that $\mathscr{R}_\gamma$ can be identified with an appropriate space of geometrical objects on $\mathscr{P}$. Automorphisms of $\mathscr{P}$ induce motions in $\mathscr{R}$ and those displacements $\delta\gamma \in \mathscr{R}$ which are induced by infinitesimal automorphisms form a subspace $\mathscr{R}^c_\gamma$. Vectors from $\mathscr{R}^c_\gamma$ may be called virtual displacements *compatible with constraints*. The Golden Rule states that in a state of static equilibrium of $\gamma \in \mathscr{R}$ with respect to a given action $\mathscr{F} \in \mathscr{R}^*_\gamma$ of matter one has

$$\langle \mathscr{F}, \delta\gamma \rangle = 0 \tag{2.1}$$

for all $\delta\gamma \in \mathscr{R}^c_\gamma$.

Infinitesimal automorphisms of $\mathscr{P}$ form a Lie subalgebra T of the algebra of all vector fields on $\mathscr{P}$. When $\mathscr{R}_\gamma$ is identified with some space of geometrical objects on $\mathscr{P}$ then

$$\mathscr{R}^c_\gamma = \{ \mathrm{L}_\mathrm{X}\gamma : \mathrm{X} \in \mathrm{T} \},$$

where L_X denotes Lie derivative.

Matter can be distributed on $\mathscr{P}$ smoothly, or it can be concentrated on a submanifold $\mathscr{K}$ of $\mathscr{P}$. The latter case is more general (since we can take in particular for $\mathscr{K}$ an open subset of $\mathscr{P}$), and assuming that matter is regularly distributed on $\mathscr{K}$ (2.1) can be written as

$$\int_{\mathscr{K}} \langle \mathscr{F}, \mathrm{L}_\mathrm{X}\gamma \rangle = 0, \qquad \mathrm{X} \in \mathrm{T} \tag{2.2}$$

where $\mathscr{F}$ is some field of densities of geometrical objects (dual to those in $\mathscr{R}_\gamma$) defined on $\mathscr{K}$. To avoid inconsistencies we shall always assume that T and $\mathscr{K}$ are in such a relation that the integral (2.2) makes sense. In some cases vector fields from T can be assumed to have compact supports, and in other cases the restrictions of $\mathrm{X} \in \mathrm{T}$ to $\mathscr{K}$ will have either compact supports or vanish at infinity sufficiently fast.

3. GAUGE GEOMETRIES

Let $(\mathscr{P}, \pi, \mathrm{B}, \mathrm{G})$ be a principal bundle over B with structural Lie group G and projection $\pi : \mathscr{P} \to \mathrm{B}$. With the bundle structure fixed (constraints) a geometry γ of $\mathscr{P}$ is assumed to consist of a pair (g, ω), where g is a metric tensor on B and ω is a principal connection on $\mathscr{P}$. In order to be in agreement with the general framework of Sec. 2 we should lift g to an invariant horizontal tensor on $\mathscr{P}$. However, since final results happen to be expressible in terms of $\mathrm{K} \doteq \pi(\mathscr{K})$ only, we shall simplify our reasoning from the very beginning and assume that matter is distributed regularly on a submanifold K of B.

Let $\mathscr{C}$ be the space of all principal connections on $\mathscr{P}$. If $\omega, \omega' \in \mathscr{C}$ then $\delta\omega \doteq \omega' - \omega$ is a horizontal 1-form on $\mathscr{P}$ (i. e. $\delta\omega$ vanishes on vertical vectors) of type Ad (i. e. $\delta\omega_{pa} = \mathrm{Ad}\,(a^{-1})\delta\omega_p$ for $p \in \mathscr{P}$, $a \in \mathrm{G}$). Therefore $\mathscr{C}_\omega$

can be identified with the space of 1-forms on B with values in the associated bundle $\mathscr{P} \times_G \mathscr{G}$, where $\mathscr{G}$ is the Lie algebra of G.

Let $\mathscr{M}$ denote the space of all Riemann metrics on B. If $g, g' \in \mathscr{M}$ then $\delta g \doteq g' - g$ is a symmetric tensor of type $(0, 2)$. Therefore an element $\mathscr{F} \in \mathscr{R}_{(\omega,g)}{}^* = (\mathscr{C}_\omega)^* \oplus (\mathscr{M}_g)^*$ regularly distributed on K can be identified with a pair $(\mathscr{T}, \mathscr{J})$, where $\mathscr{T} = (\mathscr{T}^{\mu\nu})$ is a density on K with values in symmetric tensors of type $(2, 0)$, and $\mathscr{J} = (\mathscr{J}^\mu)$ is a density vector field on K with values in the associated bundle $\mathscr{P} \times_G \mathscr{G}^*$ so that

$$\langle \mathscr{F}, \delta\imath \rangle = \int_K \left(\frac{1}{2} \mathscr{T}^{\mu\nu} \delta g_{\mu\nu} + \langle \mathscr{J}^\mu, \delta\omega_\mu \rangle \right) d^m t, \tag{3.1}$$

where $x^\mu (\mu = 1, \ldots, n)$ and $t^i (i = 1, \ldots, m)$ are coordinate systems on B and K respectively.

An infinitesimal automorphism of $\mathscr{P}$ is an invariant vector field X on $\mathscr{P}$ (i. e. $[X, Z_h] = 0$ for all $h \in \mathscr{G}$, where Z_h is a fundamental vector field generated by $h \in \mathscr{G}$). If X is invariant, then $\pi_* X$ is well defined and the Golden Rule (2.1) reads

$$\int_K \left(\frac{1}{2} \mathscr{T}^{\mu\nu} L_{\pi_* X} g_{\mu\nu} + \langle \mathscr{J}^\mu, L_X \omega_\mu \rangle \right) = 0, \qquad X \in T, \tag{3.2}$$

where T is the Lie algebra of all invariant vector fields X such that π (supp X) is compact.

To investigate consequences of (3.2) we observe that $T = T_V \oplus T_H$, where T_V (resp. T_H) is the space of all vertical (resp. horizontal) vector fields from T. If $X \in T_V$ then $\pi_* X = 0$ and X can be identified with a section χ of $\mathscr{P} \times_G \mathscr{G}$ so that $X(p) = Z_{h_p}(p)$, where $p \,.\, h_p = \chi(p)$. With such an identification one has

$$(L_X \omega)(\xi) = \xi^\mu D_\mu \chi, \tag{3.3}$$

where D_μ denotes the covariant derivative with respect to ω. Replacing χ by $\alpha\chi$, where α is any function on B vanishing on K, we get from (3.2) and (3.3) (*see* [*3*]):

$$\int_K \langle d\alpha(\mathscr{J}), \chi \rangle \, d^m t = 0, \tag{3.4}$$

and taking into account arbitrariness of χ and α we deduce that the vector $\mathscr{J}^\mu$ is tangent to K so that there exists a vector density $j = (j^i)$ such that

$$\mathscr{J}^\mu = x^\mu_i j^i \tag{3.5}$$

where $x^\mu_i = \partial x^\mu / \partial t^i$. From (3.3-3.5) we have

$$\int \langle j^i, D_i \chi \rangle \, d^m t = 0$$

for all χ. Since $\langle j^i, D_i \chi \rangle = \partial_i (\langle j^i, \chi \rangle) - \langle D_i j^i, \chi \rangle$, and owing to the arbitrariness of χ on the boundary ∂K of K we deduce that j^i on ∂K is tangent to ∂K. On the other hand, since χ is arbitrary in the interior of K, we conclude that

$$D_i j^i = 0 \tag{3.6}$$

It remains to consider (3.2) for $X \in T_H$. Every such X is a horizontal lift $\lambda\zeta$ of a vector field $\zeta = \pi_* X$ on B. Since $(L_{\lambda\zeta}\omega)(\lambda\xi) = \Omega(\xi, \zeta)$, where $\Omega = D\omega$ is the curvature 2-form, and since $L_\zeta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$, it follows that

$$\int_K (\mathcal{T}^{\mu\nu}\nabla_\mu\zeta_\nu + \langle \mathcal{J}^\mu, \Omega_{\mu\nu} \rangle \zeta^\nu) d^m t = 0 \tag{3.7}$$

Replacing ζ by $\alpha\zeta$ as above and taking into account symmetry of $\mathcal{T}^{\mu\nu}$ we deduce that there exists a symmetric tensor density t^{ij} on K such that

$$\mathcal{T}^{\mu\nu} = t^{ij} x_i^\mu x_j^\nu \tag{3.8}$$

The term $\mathcal{T}^{\mu\nu}\partial_\mu\zeta_\nu$ in (3.7) can be now transformed into $\partial_i(t^{ij}x_j^\nu\zeta_\nu) + \zeta^\nu\partial_i(t^{ij}x_j^\nu)$ and, since ζ is arbitrary on ∂K, it follows that t^{ij} on ∂K is tangent to ∂K. On the other hand arbitrariness of ζ in the interior of K leads to

$$\partial_i(t^{ij}x_j^\nu) + \Gamma^\nu_{\mu\sigma} t^{ij} x_i^\mu x_j^\sigma + \langle \mathcal{J}^\mu, \Omega^\nu_\mu \rangle = 0 \tag{3.9}$$

Let $g_{ij} = g_{\mu\nu} x_i^\mu x_j^\nu$ be the induced metric on K. The Levi-Civita connection of g_{ij} can be easily found to be

$$\Gamma_{ij,k} = \Gamma_{\mu\nu,\sigma} x_i^\mu x_j^\nu x_k^\sigma + g_{\mu\nu} x_{ij}^\mu x_k^\nu .$$

After contracting (3.9) with $g_{\mu\lambda} x_k^\lambda$ we find

$$\nabla_i t^{ij} + \langle j^i, \Omega_i^j \rangle = 0 \tag{3.10}$$

where $\Omega_{ij} \doteq \Omega_{\mu\nu} x_i^\mu x_j^\nu$ is the restriction of $\Omega_{\mu\nu}$ to K and ∇_i is the Levi-Civita connection of (K, g_{ij}). It follows that t^{ij} and j_i can be interpreted as internal energy-momentum tensor and current densities on K.

To summarize our discussion: *the following conditions i) and ii) are necessary for matter regularly distributed on a submanifold* K *of* B *to be in equilibrium with geometry represented by* (g, ω):

i) there exists a symmetric tensor density t^{ij} *and a density* j^i *with values in* $K \times_G \mathcal{G}^*$ *such that*

$$D_i j^i = 0 \tag{3.11}$$

$$\partial_i(t^{ij}x_j^\sigma) + \Gamma^\sigma_{\mu\nu} t^{ij} x_i^\mu x_j^\nu + \langle j^i, \Omega^\sigma_\mu \rangle x_i^\mu = 0 , \tag{3.12}$$

and, in particular,

$$\nabla_i(t^{ij}) + \langle j^i, \Omega_i^j \rangle = 0 , \tag{3.13}$$

ii) if K *has a boundary* ∂K *then* t^{ij} *and* j^i *are tangent to* ∂K *on* ∂K.

Remarks. 1) To make it easier to compare our results with those obtained by different methods we give some explicit expressions. Let

$e_\alpha (\alpha = 1, \ldots, p)$ be a basis in $\mathcal{G}$ and let $[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma$, $C^\gamma_{\alpha\beta}$ being the structure constants for $\mathcal{G}$. We also fix a section σ of $\mathcal{P}$ (gauge) to introduce $A_\mu = A^\alpha_\mu e_\alpha \doteq -\sigma_* \omega_\mu$, $F_{\mu\nu} = F^\alpha_{\mu\nu} e_\alpha \doteq -\sigma^* \Omega_{\mu\nu}$, and let $j^i = j^i_\alpha e^*_\alpha$, where e^*_α is the dual basis in $\mathcal{G}^*$. Then

$$\langle j^i, \Omega^\sigma_\mu \rangle = j^i_\alpha F^{\alpha\sigma}_\mu ,$$

$$(D_i j^i)_\alpha = \partial_i j^i_\alpha - C^\gamma_{\alpha\beta} A^\beta_i j^i_\gamma$$

and

$$\nabla_i l^{ij} = \partial_i l^{ij} + \Gamma^j_{ik} l^{ik} .$$

2) When we talk about densities we always mean densities with respect to coordinate systems on K i. e.

$$j^{i'} = | \partial t/\partial t' | . (\partial t^{i'}/\partial t^i) j^i ,$$

$$\mathcal{E}'^{\mu\nu} = | \partial t/\partial t' | \mathcal{E}^{\mu\nu}, \qquad \text{etc.}$$

3) When K has a boundary ∂K and a local coordinate system (t^i) on K is chosen in such a way that t^1 = const represents points on ∂K, then *ii*) means that $j^1 \equiv 0$ and $l^{1i} \equiv 0$, $i = 1, \ldots, m$, on ∂K.

4) When dim K = 1 we put $m \doteq l^{11}$ and $q \doteq j^1$. One can always choose a parameter $t^1 = s$ on K in such a way that $g_{11} \equiv 1$ (proper time parametrization). Then (3.13) implies that m = const and (3.11), (3.12) read

$$m(\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) + q_\alpha F^{\sigma\alpha}_\nu \dot{x}^\nu = 0,$$

$$\dot{q}_\alpha - C^\gamma_{\alpha\beta} q_\gamma \dot{x}^\mu A^\beta_\mu = 0,$$

which are known as Wong's equations [4] [7].

5) When dim K = 2, and *assuming* l^{ij} *to have determinant* -1, one can always choose coordinates (τ, σ) on K such that $l^{ij} = \eta^{ij}$ is constant and diagonal e. g. $\eta_{ij} = \text{diag}(2\pi, 2\pi)$. Then with $\dot{x}^\mu \doteq \partial x^\mu/\partial\tau$, $x'^\mu \doteq \partial x^\mu/\partial\sigma$, $\rho \doteq l_{1i} j^i$ and $J \doteq l_{2i} j^i$, the equations (3.11) and (3.12) become

$$\ddot{x}^\sigma - x''^\sigma + \Gamma^\sigma_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu - x'^\mu x'^\nu) - 2\pi F_{\mu\nu}(\dot{x}^\nu \rho - x'^\nu J) = 0,$$

$$(\partial\rho/\partial\tau) - (\partial J/\partial\sigma) = C^\gamma_{\alpha\beta} A^\beta_\mu (\dot{x}^\mu \rho - x'^\mu J).$$

These equations coincide formally with those given by Nielsen [8] but this coincidence is not exact, since although J necessarily vanishes on a boundary σ = const, we get no endpoint condition $x'^\mu = 0$ and no Cauchy data's constraints.

6) If $L_X \lambda = 0$ then X may be called a Killing vector field for λ. In our case $X = (\xi, \chi)$ is such a field if and only if

i) $L_\xi g_{\mu\nu} = 0$

ii) $D_\mu \chi + \Omega_{\mu\nu} \xi^\nu = 0$.

Every Killing vector field (ξ, χ) determines a conserved quantity (*see* [3] where point particles are discussed). Suppose K is parametrized by

$(\tau, t^2, \ldots, t^m)$ in such a way that sections $\tau = \text{const}$ are bounded. Given a Killing vector field $X = (\xi^\mu, \chi)$ define for each τ

$$P_X(\tau) = \int_{\tau=\text{const}} (\xi_\mu x_i^\mu j^{i1} + \langle j^1, \chi \rangle) d^{m-1}t .$$

Then P_X is independent of τ and may be called a momentum of K in X direction.

For an Abelian gauge field every constant χ defines a Killing vector so that $\int j^1$ is conserved, and, on the other hand, *i*) and *ii*) imply that $L_\xi g_{\mu\nu} = L_\xi \Omega_{\mu\nu} = 0$. Every such ξ determines a unique (up to an additive constant) χ such that (ξ, χ) is a Killing vector field for (g, ω).

4. COMPARISON WITH KALUZA-KLEIN APPROACH

N. K. Nielsen has derived his equations for a charged string via a Kaluza-Klein theory [8]. In that framework one considers $\mathscr{K}$ to be a submanifold of $\mathscr{P}$ and starts with an action

$$s \sim \int_{\mathscr{K}} h^{\frac{1}{2}} d^m t \tag{4.1}$$

where $h = |\det h_{ij}|$, h_{ij} being a metric on $\mathscr{K}$ induced by metric

$$g_{AB}(A, B = 1, \ldots, n+p) \quad \text{on} \quad \mathscr{P}.$$

To define g_{AB} it is convenient to choose an orthonormal frame $(e_m)_{m=1,\ldots,n}$ on B, and a Lie algebra basis $e_\alpha(\alpha = 1, \ldots, p)$ in $\mathscr{G}$, to form a vielbein $\hat{e}_A$ on $\mathscr{P}$ defined by $\hat{e}_m \doteq \lambda e_m$ (λ being the horizontal lift) and $\hat{e}_\alpha = Z_{e_\alpha}$. Then g_{AB} is defined by $g_{mn} \doteq \eta_{mn}$ (diagonal and constant), $g_{\alpha\beta}$ being an invariant scalar product in $\mathscr{G}$ (assumed to exist), and $g_{\alpha m} \doteq 0$. One gets then for $\mathscr{K}$ the simple equation

$$D * d\iota = 0, \tag{4.2}$$

where $\iota : \mathscr{K} \to \mathscr{P}$ is the canonical injection: $\iota(p) = p$ for $p \in \mathscr{K}$. The $*$ denotes Hodge dual operator for $\mathscr{K}$ equipped with induced metric $h_{ij} \doteq g_{AB} x_i^A x_j^B$, and D stands for exterior covariant derivative with respect to an affine connection on $\mathscr{P}$. We observe that $d\iota$, which maps every vector tangent to $\mathscr{K}$ into itself but considered as tangent to $\mathscr{P}$, may be interpreted as a 1-form on $\mathscr{K}$ with values in the associated bundle $T\mathscr{P}|_{\mathscr{K}}$.

Equation (4.2) makes sense for any affine connection on $\mathscr{P}$, but the specific action (4.1) singles out the Levi-Civita connection Γ^B_{AC} of g_{AB}. If $(t^i)_{i=1,\ldots,m}$ are coordinates on $\mathscr{K}$, and x_i^A is defined by $\partial_i = x_i^A \hat{e}_A$, then (4.2) becomes

$$h^{ij}(\nabla_i x_j^A + \Gamma^A_{BC} x_i^B x_j^C) = 0, \tag{4.3}$$

where ∇_i stands for Levi-Civita connection for $(\mathscr{H}, h_{ij})$ and Γ^{A}_{BC} are given by $\Gamma^{m}_{nr} = \left\{ {m \atop nr} \right\}$ – the Levi-Civita connection for $g^{\mu\nu} \doteq \eta^{mn} e^{\mu}_{m} e^{\nu}_{n}$, $\Gamma^{\alpha}_{mn} = -\frac{1}{2}\Omega^{\alpha}_{mn}$, $\Gamma^{n}_{m\alpha} = \Gamma^{n}_{\alpha m} = \frac{1}{2}\Omega^{n}_{m\alpha}$, $\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}C^{\gamma}_{\alpha\beta}$ – the structure constants of G.

For $A = \alpha$ (4.3) gives

$$\nabla_i J^i = 0, \tag{4.4}$$

where

$$J^{i\alpha} = h^{ij} J^{\alpha}_{j} \doteq -h^{ij} x^{\alpha}_{j}, \tag{4.5}$$

and for $A = m$ we get

$$h^{ij}(\nabla_i x^{n}_{j} + \left\{ {n \atop mr} \right\} x^{m}_{i} x^{r}_{j}) + J^{i}_{\alpha} \Omega^{n}_{mr} x^{m}_{i} = 0. \tag{4.6}$$

Equations (4.4)-(4.6) have a form similar to (3.11)-(3.12) but the meaning of symbols is different. The current j^i is a vector density on $K = \pi(\mathscr{H})$ with values in $\mathscr{P} \times_G \mathscr{G}^*$ while J^{i}_{α} is a vector on $\mathscr{H}$ with values in $\mathscr{G}^*$. The « D_i » in (3.11) refers to a covariant derivative with respect to ω whereas ∇_i in (4.4) is the Levi-Civita connection. However it is enough to consider $\mathscr{H}$ as a section $\sigma : K \to \mathscr{H}$ and put

$$j^i(y) \doteq h(\sigma(y))^{\frac{1}{2}} \sigma(y) \,.\, J^i(\sigma(y)), \qquad y \in K, \tag{4.7}$$

to make (4.4) and (3.11) exactly to coincide. Similarly, with

$$l^{ij}(y) \doteq h(\sigma(y))^{\frac{1}{2}} h^{ij}(\sigma(y)), \qquad y \in K,$$

(3.12) and (4.6) coincide. We note that it follows by the very definition that

$$h_{ij} = g_{ij} + g_{\alpha\beta} J^{\alpha}_{i} J^{\beta}_{j} \tag{4.8}$$

and also that

$$-J_i = \omega_i - \tilde{\omega}_i,$$

where $\tilde{\omega}$ is a unique flat connection with respect to which $\mathscr{H}$ is parallel. It follows that Nielsen's equations (4.2) constitute a very special case of more general formulas (3.11)-(3.12) characterized by the fact that there exists a current J_i such that

$$\begin{aligned} &i) && \tilde{\omega} \doteq \omega + J \quad \text{is locally flat,} \\ &ii) && l^{ij} = h^{\frac{1}{2}} h^{ij}, \\ &iii) && j^i = h^{\frac{1}{2}} \sigma \,.\, J^i, \end{aligned} \tag{4.10}$$

where h_{ij} is given by (4.8) and σ is some section over K which is parallel with respect to $\tilde{\omega}$. One can check then energy-momentum conservation formula (3.13) by explicit calculation from (4.10 *ii*), (4.8), (4.4). The end-point condition for $x(t^i)$ follows also from (4.8). On the other hand it follows from (4.10 *i*) that a conserved fluxoid characterizing the bundle $\pi^{-1}(K)$ exists for a closed string (*see* [8]).

5. G-STRUCTURES

Our definition of a G-structure differs slightly from a conventional one (*see* e. g. [*13*]) but such a modification seems to be necessary if methods of Sec. 2 are to be applied. The following definition works also quite well when one wants to describe interaction of a gravitational field with spinorial matter consistently.

Let G be a Lie group and let ρ be a homomorphism of G into GL(N). Let $\mathscr{P}$ be an N-dimensional manifold and suppose that a principal bundle $\mathscr{Q}$ over $\mathscr{P}$ with structural group G is fixed. By a G-structuration on $\mathscr{P}$ we mean a bundle homomorphism γ from $\mathscr{Q}$ into the bundle of linear frames over $\mathscr{P}$, such that

$$\gamma(qa) = \gamma(q)\rho(a), \qquad q \in \mathscr{Q}, \qquad a \in \mathrm{G}.$$

Let $\mathscr{R}$ denotes the space of all such homomorphisms. If $\gamma, \gamma' \in \mathscr{R}$ then $\gamma'(q) = \gamma(q)\Lambda(q)$, where $\Lambda : \mathscr{Q} \to \mathrm{GL(N)}$ satisfies

$$\Lambda(qa) = \rho(a^{-1})\Lambda(q)\rho(a).$$

It follows that $\mathscr{R}_\gamma$ can be identified with the space of sections of the associated bundle $\mathscr{Q} \times_{\mathrm{G}} gl(\mathrm{N})$ corresponding to the representation $\mathrm{Ad}\rho$ of $\mathscr{G}$ in $gl(\mathrm{N})$.

Every automorphism Φ of $\mathscr{Q}$ induces a map ϕ of $\mathscr{R}$ by

$$(\phi\gamma)(q) = \tilde{\Phi}^{*}\gamma(\Phi^{-1}q), \tag{5.1}$$

where $\tilde{\Phi}$ denotes the induced map $\mathscr{P}$. To describe infinitesimal automorphisms and their action on $\mathscr{R}$ it is convenient to fix a section σ of $\mathscr{Q}$ and introduce vielbein $e_{\mathrm{A}}(p) \doteq \gamma(\sigma(p))_{\mathrm{A}}$, $\mathrm{A} = 1, \ldots, \mathrm{N}$. Then $\mathscr{R}_\gamma$ can be identified with space of functions $\Lambda : \mathscr{P} \to gl(\mathrm{N})$. Infinitesimal automorphisms of $\mathscr{Q}$ form a Lie algebra T of invariant vector fields on $\mathscr{Q}$. If X is such a field then $\xi = \pi_{*}\mathrm{X}$ is well defined, and there exists a unique invariant vector field Y such that $\mathrm{Y}(\sigma(p)) = (\sigma_{*}\xi)(\sigma(p))$. (In other words Y is a horizontal projection of X with respect to the flat connection induced by σ). Therefore one can split T into $\mathrm{T} = \mathrm{T_V} \oplus \mathrm{T_H}$ (This splitting is not a natural one and depends on σ), and it is easy to see that (5.1) implies that $\mathscr{R}^c_\gamma$ consists of $\delta\Lambda$-*s* of two types

$$i) \qquad \delta_{\mathrm{V}}\Lambda = \rho'(v), \qquad v : \mathscr{P} \to \mathscr{G},$$

$$ii) \qquad \delta_{\mathrm{H}}\Lambda^{\mathrm{A}}_{\mathrm{B}} = [\xi, e_{\mathrm{B}}]^{\mathrm{A}}, \tag{5.2}$$

where ρ' is the derived representation of $\mathscr{G}$ in GL(N), and ξ is a vector field on $\mathscr{P}$. Both v and ξ will be assumed to have compact supports. It is clear from (5.2) that we can restrict ourselves to a case when $\mathrm{G} = \rho(\mathrm{G})$ is a subgroup of GL(N) without loosing generality.

Let $\mathscr{K}$ be an m-dimensional submanifold of $\mathscr{P}$ and assume that matter is regularly distributed on $\mathscr{K}$ and is represented by a density

$$\mathscr{F} : \mathscr{K} \rightarrow g\ell(\mathrm{N})^* \cong g\ell(\mathrm{N}).$$

Then the Golden Rule states that in equilibrium of matter $\mathscr{F}$ and G-structure γ (represented by Λ) we have

$$\int_{\mathscr{K}} \mathscr{F}^{\mathrm{B}}_{\mathrm{A}} \delta\Lambda^{\mathrm{A}}_{\mathrm{B}} = 0, \qquad \delta\Lambda \in \mathscr{R}^c_\gamma \tag{5.3}$$

From (5.2 i) one gets immediately

$$\mathscr{F}^{\mathrm{B}}_{\mathrm{A}} v^{\mathrm{A}}_{\mathrm{B}} = 0, \qquad v \in \mathscr{G}, \tag{5.4}$$

and ii) implies that there exists vector density f^i_{A} on $\mathscr{K}$ such that $\mathscr{F}^{\mathrm{B}}_{\mathrm{A}} = x^{\mathrm{B}}_i f^i_{\mathrm{A}}$, where x^{B}_i is the component of a vector ∂_i, tangent to the coordinate line t^i, with respect to e_{B}. We also find that

$$\begin{aligned} &p)\ \ f^i_{\mathrm{A}} \text{ is tangent to } \partial\mathscr{K} \text{ on } \partial\mathscr{K} \quad \text{for} \quad \mathrm{A} = 1, \ldots, \mathrm{N} \\ &pp)\ \ \partial_i f^i_{\mathrm{A}} - \mathrm{E}^{\mathrm{C}}_{\mathrm{BA}} x^{\mathrm{B}}_i f^i_{\mathrm{C}} = 0, \end{aligned} \tag{5.5}$$

where $\mathrm{E}^{\mathrm{C}}_{\mathrm{AB}}$ are defined by

$$[e_{\mathrm{A}}, e_{\mathrm{B}}] = \mathrm{E}^{\mathrm{C}}_{\mathrm{AB}} e_{\mathrm{C}}. \tag{5.6}$$

SPECIAL CASES.

1) **Orthogonal structures.**

Suppose G is a group of all linear transformations preserving nondegenerate symmetric matrix η_{AB}. The condition (5.4) gives then $\mathscr{F}^{\mathrm{BA}} = \mathscr{F}^{\mathrm{AB}}$, where $\mathscr{F}^{\mathrm{AB}} = \mathscr{F}^{\mathrm{A}}_{\mathrm{C}}\eta^{\mathrm{CB}}$, and it follows that there exists a symmetric tensor density f^{ij} on $\mathscr{K}$ such that

$$f^i_{\mathrm{A}} = \eta_{\mathrm{AB}} x^{\mathrm{B}}_j f^{ij},$$

f^{ij} being tangent to $\partial\mathscr{K}$ on $\partial\mathscr{K}$. Let $\Gamma^{\mathrm{A}}_{\mathrm{BC}}$ be the coefficients of the Levi-Civita connection for $(\mathscr{P}, \eta_{\mathrm{AB}})$. Since Γ has no torsion we have

$$\Gamma^{\mathrm{C}}_{\mathrm{AB}} - \Gamma^{\mathrm{C}}_{\mathrm{BA}} = \mathrm{E}^{\mathrm{C}}_{\mathrm{AB}},$$

and owing to (5.4) we get

$$\mathrm{E}^{\mathrm{C}}_{\mathrm{AB}} \mathscr{F}^{\mathrm{B}}_{\mathrm{C}} = -\Gamma^{\mathrm{C}}_{\mathrm{BA}} \mathscr{F}^{\mathrm{B}}_{\mathrm{C}},$$

so that (5.5) becomes

$$\partial_i(f^{ij} x^{\mathrm{C}}_j) + \Gamma^{\mathrm{C}}_{\mathrm{AB}} f^{ij} x^{\mathrm{A}}_i x^{\mathrm{B}}_j = 0 \tag{5.7}$$

It is convenient to introduce tensor $\mathrm{T}^{ij} \doteq h^{-\frac{1}{2}} f^{ij}$, where $h_{ij} \doteq \eta_{\mathrm{AB}} x^{\mathrm{A}}_i x^{\mathrm{B}}_j$ is the induced metric on $\mathscr{K}$. Then (5.7) reads

$$\nabla_i(\mathrm{T}^{ij} x^{\mathrm{C}}_j) + \mathrm{T}^{ij}\Gamma^{\mathrm{C}}_{\mathrm{AB}} x^{\mathrm{A}}_i x^{\mathrm{B}}_j = 0, \tag{5.8}$$

where ∇_i is the Levi-Civita connection for $(\mathcal{K}, h_{ij})$. By contraction with x_{jA} it follows from (5.8) that

$$\nabla_i T^{ij} = 0, \tag{5.9}$$

and a particular, natural, solution of (5.9) is $T^{ij} = h^{ij}$. In such a case (5.8) coincides with (4.3) (we note that (5.8) remains true when vielbein indices A, B, C, ... are replaced with indices corresponding to a coordinate system on $\mathcal{P}$). It is only for dim $\mathcal{K} = 1$ that (5.8) determines dynamics of $\mathcal{K}$ completely (geodesic principle). When dim $\mathcal{K} > 1$ one has to specify in addition an internal dynamics of $\mathcal{K}$ i. e. to single out a particular, conserved, energy-momentum tensor T^{ij} on $\mathcal{K}$.

2) **Supergravity.**

Supergravity has been formulated [*9*] as a constrained Lorentz structure on superspace. The relevant group G can be described here as follows: let $\gamma_m = (\gamma^{\mu}_{m\nu})$ be a fixed set of real γ-matrices ($m, n = 0, 1, 2, 3$, $\mu, \nu = 1, 2, 3, 4$) satisfying $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$, $\eta = \mathrm{diag}(-1, +1, +1, +1)$, and let $C = (C_{\mu\nu})$ be a fixed charge conjugation matrix, so that $C\gamma_m$ are symmetric; then G consists of all pairs of real 4×4 matrices (Λ, A) satisfying

i) $$\Lambda^T \eta \Lambda = \eta$$
ii) $$A^T C A = C$$
iii) $$A \gamma_m A^{-1} = \Lambda^n_m \gamma_n .$$

It is evident that G is isomorphic to SL(2, $\mathbb{C}$), and for its Lie algebra we have

$$\dot{A} = \frac{1}{2} \dot{\Lambda}^{mn} \Sigma_{mn}, \tag{5.10}$$

where

$$\Sigma_{mn} = \frac{1}{4} [\gamma_m, \gamma_n]. \tag{5.11}$$

$\mathcal{P}$ is now a supermanifold of dimension (4.4) (*see* [*14*] for relevant definitions), and G is considered to be a subgroup of GL(4, 4). Applying methods developed in this section to the case of a point particle we find that a 1-dimensional distribution $\mathcal{K}$ in $\mathcal{P}$ has to satisfy

$$\begin{aligned} &a) \quad \dot{p}_A - \dot{x}^B \Gamma^C_{BA} p_C + \dot{x}^B T^C_{AB} p_C = 0, \\ &b) \quad (-1)^B v^A_B p_A \dot{x}^B = 0 , \end{aligned} \tag{5.12}$$

where $A = (m, \mu)$, p_A is an even density defined on $\mathcal{K}$, and Γ^C_{AB} and T^C_{AB} are, respectively, coefficients of a G-connection and its torsion. We remark that now the order of factors is relevant in (5.12). The relations (5.12 *b*) can be easily solved owing to (5.10) so that one gets

b') $$p_m \dot{x}_n - p_n \dot{x}_m = \Sigma_{mn}{}^{\mu}{}_{\nu} p_{\mu} \dot{x}^{\nu} .$$

Since $C\Sigma_{mn}$ and $C\gamma_5\Sigma_{mn}$ are symmetric and $\{ \dot{x}^\mu, \dot{x}^\nu \} = 0$, a particular solution of (b') can be written as

$$p_m = \mathcal{M}\dot{x}_m,$$
$$p_\mu = (C(a+b\gamma_5))_{\mu\nu}\dot{x}^\nu,$$

in which case both sides of (b') vanish separately. Here $\mathcal{M}$, a, b are arbitrary functions of the parameter t, with values in the even part of a Banach-Grassmann algebra [*14*]. One can adjust parameter t by demanding the number part of M to be constant. In case of a super-symmetric superspace [*14*] one can use a local coordinate system (x^a, θ^α) such that the vielbein $\{ e_A \}$ becomes

$$e_m = \delta^a_m \partial_a,$$
$$e_\mu = \delta^\alpha_\mu \partial_\alpha - \frac{1}{2} \gamma^a_{\mu\nu} \delta^\nu_\beta \theta^\beta \partial_a.$$

Then (5.12 a) reads

$$\left.\begin{aligned} p_a &= \text{const.}, \\ p_\alpha + \frac{1}{2} \gamma^a_{\alpha\beta} \theta^\beta p_a &= \text{const.}, \end{aligned}\right\} \tag{5.13}$$

where

$$\left.\begin{aligned} p_a &= M\dot{x}_a, \\ p_\alpha &= (C(a + b\gamma_5))_{\alpha\beta}\dot{\theta}^\beta + \frac{1}{2} M\gamma^a_{\alpha\beta}\theta^\beta \dot{x}_a. \end{aligned}\right\} \tag{5.14}$$

The equations (5.13) give conservation of momentum p_a and spinorial charge q_α

$$q_\alpha \equiv (C(a + b\gamma_5))_{\alpha\beta}\dot{\theta}^\beta + M\gamma^a_{\alpha\beta}\dot{x}_a\theta^\beta.$$

The two conservation laws can be also deduced by a reasoning similar to that of Remark 5, Section 3, the momentum conservation being a consequence of translational invariance while spinorial charge conservation follows from invariance under supertranslation. The resulting equations contain those obtained in [*15*] from a Lagrangian based on a line element. It is not clear whether other solutions of (5.12 b') can be of physical interest. It is also to be observed that the coefficient a and b need not be constant along the trajectory.

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