Two-spinors and Einstein-Cartan-Maxwell-Dirac fields*

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Abstract

We show that a complex vector bundle $\boldsymbol{S} \to \boldsymbol{M}$, where $\boldsymbol{M}$ is a 4-dimensional real manifold and the fibres of $\boldsymbol{S}$ are 2-dimensional, yields in a natural way *all* structures which are needed in order to formulate a (classical) theory of Einstein-Cartan-Maxwell-Dirac fields. Namely, all needed bundles and their fibre structures follow from functorial constructions with no further assumptions. Any considered object which is not a functorial construction is taken to be a field. This is true even for coupling constants, which arise as constant sections of real line bundles derived form $\boldsymbol{S}$.

In the above said context we also discuss to what extend one can give a formulation which is not singular in the case of a degenerate vierbein.

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Contents

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1 Preliminaries

We assume that the reader is familiar with the basic notions concerning linear connections on real and complex vector bundles.

If X is any set and $f : X \to \mathbb{C}$ then we denote by $\bar{f}$ the conjugate map, $\bar{f}(x) := \overline{f(x)}$.

If $\boldsymbol{V}$ is a finite-dimensional complex vector space then we shall denote by $\boldsymbol{V}^{\mathrm{F}}$ and $\boldsymbol{V}^*$ its complex and real dual spaces, respectively. Moreover we shall denote by $\boldsymbol{V}^{\overline{\mathrm{F}}}$ the antidual space (i.e. the space of all antilinear maps $\boldsymbol{V} \to \mathbb{C}$), and by $\overline{\boldsymbol{V}} := (\boldsymbol{V}^{\mathrm{F}})^{\overline{\mathrm{F}}}$ the conjugate space (then $\overline{\boldsymbol{V}}^{\mathrm{F}} \cong \boldsymbol{V}^{\overline{\mathrm{F}}}$). We have the (conjugation) anti-isomorphisms $\boldsymbol{V}^{\mathrm{F}} \to \boldsymbol{V}^{\overline{\mathrm{F}}}$ and $\boldsymbol{V} \to \overline{\boldsymbol{V}}$.

Let (ζ_a) be a basis of $\boldsymbol{V}$, $a = 1, \dots, n$, and (z^a) the dual basis of $\boldsymbol{V}^{\mathrm{F}}$. Then we have the conjugate bases $(\bar{\zeta}_{\dot{a}}) := (\overline{\zeta_a})$ of $\overline{\boldsymbol{V}}$ and $(\bar{z}^{\dot{a}}) := (\overline{z^a})$ of $\boldsymbol{V}^{\overline{\mathrm{F}}}$. If $v = v^a \zeta_a$ and $\lambda = \lambda_a z^a$ then $\bar{v} = \bar{v}^{\dot{a}} \bar{\zeta}_{\dot{a}}$ and $\bar{\lambda} = \bar{\lambda}_{\dot{a}} \bar{z}^{\dot{a}}$ with $\bar{v}^{\dot{a}} = \overline{v^a}$, $\bar{\lambda}_{\dot{a}} = \overline{\lambda_a}$.

The (real) differential of a function $f : \boldsymbol{V} \to \mathbb{R}$ is a 1-form $df : \boldsymbol{V} \to \boldsymbol{V}^* \subset \boldsymbol{V}^{\mathrm{F}} \oplus \boldsymbol{V}^{\overline{\mathrm{F}}}$. In coordinates we write $df = \frac{\partial f}{\partial z^a} dz^a + \frac{\partial f}{\partial \bar{z}^{\dot{a}}} d\bar{z}^{\dot{a}}$, namely we formally consider z^a and $\bar{z}^{\dot{a}}$ as real independent coordinates.

Conjugation can be naturally extended to tensor products of the above spaces with any number of factors. If τ is a tensor then $\bar{\tau}$ has dotted indices in the place of non-dotted indices of τ, and vice-versa.

A tensor $w \in \boldsymbol{V} \otimes \overline{\boldsymbol{V}}$ is said to be Hermitian if $\bar{w} = w^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes trasposition. In coordinates this means $\bar{w}^{\dot{b}a} = w^{a\dot{b}}$. We have the *real* decomposition $\boldsymbol{V} \otimes \overline{\boldsymbol{V}} = \boldsymbol{HV} \oplus i\,\boldsymbol{HV}$ into Hermitian and anti-Hermitian subspaces.

If $\boldsymbol{V}$ is 1-dimensional then $\boldsymbol{V} \cong \boldsymbol{V}^{1/2} \otimes \boldsymbol{V}^{1/2}$ where $\boldsymbol{V}^{1/2}$ is its *square root* space, which is unique up to an isomorphism. Similarly one defines $\boldsymbol{V}^{1/p}$ for $p \in \mathbb{N}$, hence also $\boldsymbol{V}^{q/p}$ for $p, q \in \mathbb{N}$. If we set $\boldsymbol{V}^{-1} := \boldsymbol{V}^{\mathrm{F}}$, the power of a 1-dimensional complex vector space is naturally defined for all rational exponents.

Note that, again if $\boldsymbol{V}$ is 1-dimensional, the Hermitian subspace $\boldsymbol{HV} \subset \boldsymbol{V} \otimes \overline{\boldsymbol{V}}$ is a 1-dimensional real oriented vector space, and $\boldsymbol{V} \cong \mathbb{C} \otimes (\boldsymbol{HV}^+)^{1/2}$ (in the real case, roots of positive semi-spaces are naturally defined for any selected orientation).

2 Two-spinor algebra

2.1 Two-spinor space

Throughout this section $\boldsymbol{S}$ will stand for a 2-dimensional complex vector space, called the *space of 2-spinors* (see also [PR84, PR88, W84, HT85]). We have the real splitting

$$\boldsymbol{S} \otimes \overline{\boldsymbol{S}} = \boldsymbol{H} \oplus i\boldsymbol{H} ,$$

where $\boldsymbol{H} := H(\boldsymbol{S}) \subset \boldsymbol{S} \otimes \overline{\boldsymbol{S}}$ is the Hermitian subspace.

We set $\boldsymbol{\Lambda}^2 := \wedge^2 \boldsymbol{S}$. We identify $\boldsymbol{\Lambda}^{-2} := \boldsymbol{\Lambda}^{2\mathrm{F}}$ with $\wedge^2 \boldsymbol{S}^{\mathrm{F}}$ through the rule[1] $\omega(s \wedge s') := \frac{1}{2}\omega(s, s')$, $\forall\, \omega \in \wedge^2 \boldsymbol{S}^{\mathrm{F}}$, $s, s' \in \boldsymbol{S}$, where $s \wedge s' := \frac{1}{2}(s \otimes s' - s' \otimes s)$.

If $\omega \neq 0$ then it has a unique 'inverse' or 'dual' element $\omega^{-1} \equiv \omega^{\mathrm{F}}$ such that $\omega(\omega^{-1}) = 1$. We indicate by $\omega^\flat : \boldsymbol{S} \to \boldsymbol{S}^{\mathrm{F}}$ the linear map given by $\langle \omega^\flat(s), t \rangle := \omega(s, t)$, and by $\omega^\# : \boldsymbol{S}^{\mathrm{F}} \to \boldsymbol{S}$ the linear map given by $\langle \mu, \omega^\#(\lambda) \rangle := \omega^{-1}(\lambda, \mu)$. Note that $\omega^\# = -(\omega^\flat)^{-1}$.

The Hermitian subspace of $\boldsymbol{\Lambda}^2 \otimes \overline{\boldsymbol{\Lambda}}^2$ is a real vector space with a distinguished orientation; its positive semi-space will be indicated by $\boldsymbol{L}^2$, thus $\boldsymbol{\Lambda}^2 \otimes \overline{\boldsymbol{\Lambda}}^2 = \mathbb{C} \otimes \boldsymbol{L}^2$. Moreover we have the square root semi-space $\boldsymbol{L}$, characterized by $\boldsymbol{L}^2 \cong \boldsymbol{L} \otimes \boldsymbol{L}$.

In general, if r is a rational number then we define $\boldsymbol{\Lambda}^r := (\boldsymbol{\Lambda}^2)^{r/2}$, hence $\boldsymbol{\Lambda}^r \otimes \overline{\boldsymbol{\Lambda}}^r = \mathbb{C} \otimes \boldsymbol{L}^r$. In particular we obtain ($r = 1$) the square root $\boldsymbol{\Lambda}$ of $\boldsymbol{\Lambda}^2$.

[1] This contraction, defined in such a way to respect the usual conventions in two-spinor literature, corresponds to half standard tensor algebra contraction and 1/4 standard exterior algebra contraction.

2.2 Bases

Let (ζ_A) be a basis of $\boldsymbol{S}$. We have the the following induced bases:

- The basis $(\sigma_\lambda := \sigma_\lambda{}^{AB'}\zeta_A \otimes \bar{\zeta}_{B'})$ of $\boldsymbol{H}$, where the $(\sigma_\lambda{}^{AB'})$'s are the Pauli matrices.
- The dual basis $(s^\lambda := s^\lambda{}_{AB'} z^A \otimes \bar{z}^{B'})$ of $\boldsymbol{H}^* \subset \boldsymbol{S}^{\mathrm{F}} \otimes \overline{\boldsymbol{S}}^{\mathrm{F}}$, where $(s^\lambda{}_{AB'})$ is half the transposed λ-th Pauli matrix.
- The basis $(\hat{\sigma}_\lambda := \sigma_\lambda{}^A{}_B \zeta_A \otimes z^B)$ of $\mathrm{End}(\boldsymbol{S}) \cong \boldsymbol{S} \otimes \boldsymbol{S}^{\mathrm{F}}$, where $(\hat{\sigma}_\lambda) := (\sigma_\lambda)$.
- The basis $\varepsilon := \varepsilon_{AB} z^A \wedge z^B$ of $\boldsymbol{\Lambda}^{-2}$.
- The dual basis $\varepsilon^{\mathrm{F}} := \varepsilon^{-1} = \varepsilon^{AB}\zeta_A \wedge \zeta_B$ of $\boldsymbol{\Lambda}^2$.
- The basis $|\varepsilon|^r := (\varepsilon \otimes \bar{\varepsilon})^{r/2}$ of $\boldsymbol{L}^r$, with dual basis $|\varepsilon|^{-r} = |\varepsilon^{-1}|^r$.

We shall also consider the mutually dual bases

$$(\tau_\lambda := \tfrac{1}{\sqrt{2}}\sigma_\lambda)\ , \qquad (t^\lambda := \sqrt{2}\, s^\lambda)\ ,$$

whose matrices are, respectively, Pauli matrices divided by $\sqrt{2}$ and transposed Pauli matrices divided by $\sqrt{2}$. We have

$$\hat{\sigma}_\lambda \circ \hat{\sigma}_\mu = (\eta_{\lambda\mu\nu} + i\,\epsilon_{0\lambda\mu\nu})\delta^{\nu\rho}\hat{\sigma}_\rho := \Sigma_{\lambda\mu\nu}\delta^{\rho\nu}\hat{\sigma}_\rho := \Sigma_{\lambda\mu}{}^\rho\hat{\sigma}_\rho\ ,$$

where

$$\eta_{\lambda\mu\nu} := \delta_{\lambda\mu}\delta^0_\nu + \delta_{\nu\lambda}\delta^0_\mu + \delta_{\mu\nu}\delta^0_\lambda - 2\delta^0_\lambda\delta^0_\mu\delta^0_\nu$$

is a totally symmetric symbol. Note that under exchange of two nearby indices $\Sigma_{\lambda\mu\rho}$ transforms into its complex conjugate. By convention, in the symbols $\eta_{\lambda\mu\nu}$, $\varepsilon_{\lambda\mu\nu\rho}$ and $\Sigma_{\lambda\mu\nu}$, indices are raised and lowered via $\delta_{\lambda\mu}$ and $\delta^{\lambda\mu}$. Some further useful formulas:

$$\begin{aligned}\eta_{\lambda\mu\nu}\,\eta^{\lambda\mu\rho} &= 2(\delta^\rho_\nu - \delta^\rho_0\,\delta^0_\nu)\ ,\\ \varepsilon_{0\lambda\mu\nu}\,\varepsilon^{0\lambda\mu\rho} &= 2(\delta^\rho_\nu + \delta^\rho_0\,\delta^0_\nu)\ ,\\ \Sigma_{\lambda\mu\nu}\,\bar{\Sigma}^{\lambda\mu\rho} &= 4\delta^\rho_\nu\ .\end{aligned}$$

2.3 Conformal Lorentz structure

We are going to show that each element of $\boldsymbol{L}^{-2}$ can be viewed as a Lorentz metric on $\boldsymbol{H}$. This means that there is a natural conformal Lorentz structure on $\boldsymbol{H}$.

The basis expressions of an element $\boldsymbol{\phi} \in \boldsymbol{L}^{-1}$ and its inverse $\boldsymbol{\phi}^\# := \boldsymbol{\phi}^{-1} \in \boldsymbol{L}$ are given by

$$\boldsymbol{\phi} = \phi\,|\varepsilon| \quad , \qquad \boldsymbol{\phi}^{-1} = \phi^{-1}\,|\varepsilon|^{-1}\ ,$$

with $\phi \in \mathrm{R}^+$. We now observe that $\boldsymbol{\phi}^2$ can be seen as a bilinear form $g_{\boldsymbol{\phi}}$ on $\boldsymbol{S} \otimes \overline{\boldsymbol{S}}$, which for decomposable tensors is given by

$$g_{\boldsymbol{\phi}}(p \otimes \bar{q}, r \otimes \bar{s}) = \phi^2\,\varepsilon(p, r)\bar{\varepsilon}(\bar{q}, \bar{s})\ .$$

Then $g_{\boldsymbol{\phi}}$ restricted to $\boldsymbol{H}$ turns out to be a Lorentz metric, actually we have the orthonormal bases

$$(\phi^{-1}\,\tau_\lambda)\ , \quad (\phi\, t^\lambda)\ .$$

We have the coordinate expression

$$g_{\boldsymbol{\phi}}(u, v) = \phi^2\,\eta_{\lambda\mu}\,u^\lambda v^\mu$$

with $u = u^\lambda \tau_\lambda$ and the like, and

$$\eta_{\lambda\mu} = \tau_\lambda{}^{AB^\cdot}\tau_\mu{}^{CD^\cdot}\varepsilon_{AC}\bar\varepsilon_{B^\cdot D^\cdot} = \tfrac{1}{2}\,\sigma_\lambda{}^{AB^\cdot}\sigma_\mu{}^{CD^\cdot}\varepsilon_{AC}\bar\varepsilon_{B^\cdot D^\cdot} = 2\delta^0_\lambda\delta^0_\mu - \delta_{\lambda\mu}\ ,$$
$$\eta^{\lambda\mu} = t^\lambda{}_{AB^\cdot}t^\mu{}_{CD^\cdot}\varepsilon^{AC}\bar\varepsilon^{B^\cdot D^\cdot} = 2\,s^\lambda{}_{AB^\cdot}s^\mu{}_{CD^\cdot}\varepsilon^{AC}\bar\varepsilon^{B^\cdot D^\cdot} = 2\delta^\lambda_0\delta^\mu_0 - \delta^{\lambda\mu}\ .$$

The Lorentz metric g_ϕ also determines, up to sign, a volume form η_ϕ on $\boldsymbol{H}$, with coordinate expression

$$\eta_\phi = \pm\tfrac{1}{4!}\,\phi^4\,\varepsilon_{\lambda\mu\nu\rho}\,t^\lambda\wedge t^\mu\wedge t^\nu\wedge t^\rho = \pm\phi^4\,t^0\wedge t^1\wedge t^2\wedge t^3\ .$$

Observe also that g_ϕ yields the map

$$g^\flat_\phi : \boldsymbol{S}\otimes\overline{\boldsymbol{S}} \to \boldsymbol{S}^{\mathrm{F}}\otimes\overline{\boldsymbol{S}}^{\mathrm{F}} : p\otimes\bar q \mapsto \phi^2\,\varepsilon^\flat(p)\otimes\bar\varepsilon^\flat(\bar q)\ ,$$

whose restriction to $\boldsymbol{H}$ is just the isomorphism $\boldsymbol{H}\to\boldsymbol{H}^*$ induced by the non-degenerate Lorentz metric. Moreover an element of $\boldsymbol{S}\otimes\overline{\boldsymbol{S}}$ can be viewed as a linear map $\overline{\boldsymbol{S}}^{\mathrm{F}}\to\boldsymbol{S}$, while an element of $\overline{\boldsymbol{S}}^{\mathrm{F}}\otimes\boldsymbol{S}^{\mathrm{F}}$ can be viewed as a linear map $\boldsymbol{S}\to\overline{\boldsymbol{S}}^{\mathrm{F}}$. So we are led to consider the map

$$\gamma_\phi := \sqrt{2}\big(\mathbf{1} + (\mathrm{g}^\flat_\phi)^{\mathrm{T}}\big) : \boldsymbol{S}\otimes\overline{\boldsymbol{S}} \to (\boldsymbol{S}\otimes\overline{\boldsymbol{S}})\oplus(\overline{\boldsymbol{S}}^{\mathrm{F}}\otimes\boldsymbol{S}^{\mathrm{F}}) \subset \mathrm{End}(\boldsymbol{S}\oplus\overline{\boldsymbol{S}}^{\mathrm{F}})\ ,$$

which on decomposable elements reads

$$\gamma_\phi(p\otimes\bar q) = \sqrt{2}\,\Big(p\otimes\bar q + \phi^2\,\bar\varepsilon^\flat(\bar q)\otimes\varepsilon^\flat(p)\Big)\ .$$

Then γ_ϕ restricted to $\boldsymbol{H}$ turns out to be a Clifford map

$$\gamma_\phi : \boldsymbol{H}\to\mathrm{End}(\boldsymbol{W})\ ,$$

where we set $\boldsymbol{W} := \boldsymbol{S}\oplus\overline{\boldsymbol{S}}^{\mathrm{F}}$. We have the basis expression

$$\gamma_\phi = \sqrt{2}\,(\tau_\lambda + \phi^2\eta_{\lambda\mu}\bar t^\mu)\otimes t^\lambda = \sqrt{2}\,(\sigma_\lambda + 2\,\phi^2\eta_{\lambda\mu}\bar s^\mu)\otimes s^\lambda\ .$$

We also consider the 'contravariant' Clifford map

$$\gamma^\#_\phi := \gamma_\phi\circ g_\phi^{-1} : \boldsymbol{H}^*\to\mathrm{End}(\boldsymbol{W})\ ,$$

with basis expression

$$\gamma^\#_\phi = \sqrt{2}\,(\phi^{-2}\,\eta^{\lambda\mu}\tau_\mu + \bar t^\lambda)\otimes\tau_\lambda\ .$$

Given $\psi = u+\alpha\in\boldsymbol{W} := \boldsymbol{S}\oplus\overline{\boldsymbol{S}}^{\mathrm{F}}$ then its 'Dirac adjoint' is $\bar\psi = \bar\alpha + \bar u \in \boldsymbol{S}^{\mathrm{F}}\oplus\overline{\boldsymbol{S}} \equiv \boldsymbol{W}^{\mathrm{F}}$.

It can be seen [CJ96] that given an 'observer', namely an element $w\in\boldsymbol{H}$ such that $g_\phi(w,w)=1$, then the quadratic map $\psi\mapsto\langle\bar\psi,\gamma_\phi(w)\psi\rangle$ is either positive or negative definite. Thus $\boldsymbol{H}$ has a natural time orientation: we call future-pointing those unit timelike elements which yield a positive quadratic map. For any basis of $\boldsymbol{S}$, the induced element τ_0 turns out to be future-oriented.

An element $\omega\in\boldsymbol{\Lambda}^{-2}$ is called *ϕ-normalized* if $\omega\otimes\bar\omega = \boldsymbol{\phi}^2$, so that $\omega = \phi\,e^{it}\,\varepsilon$, $t\in\mathrm{R}$, and $\omega^{-1} = \phi^{-1}\,e^{-it}\,\varepsilon^{-1}$. Consider the antilinear map $\mathcal{C}_\omega : \boldsymbol{W}\to\boldsymbol{W}$ given by

$$\mathcal{C}_\omega(\sqcap+\alpha) := \omega^\#(\alpha) - \bar\omega^\flat(\sqcap) = \rceil^{-\rangle\sqcup}\,(\phi^{-\infty}\,\varepsilon^\#(\alpha) - \phi\,\bar\varepsilon^\flat(\sqcap))\ .$$

Then $\mathcal{C}_\omega\circ\mathcal{C}_\omega = \mathbf{1}_{\boldsymbol{W}}$, so $\mathcal{C}_\omega$ is an anti-isomorphism.[2] We call $\mathcal{C}_\omega$ the *charge conjugation* associated with ω. Then we recover the usual fact that charge conjugation is unique up to an overall phase factor (for fixed $\boldsymbol{\phi}$).

[2]Since $\omega^\#\circ\omega^\flat = -\mathbf{1}$, if we took $\mathcal{C}_\omega(\sqcap+\alpha) := \omega^\#(\alpha) + \bar\omega^\flat(\sqcap)$ then we would get $\mathcal{C}_\omega\circ\mathcal{C}_\omega = -\mathbf{1}_{\boldsymbol{W}}$.

2.4 Lorentz structure

The fact that on $\boldsymbol{H}$ there is a natural conformal Lorentz structure, where the conformal factor belongs to the unit space $\boldsymbol{L}^{-2}$, can be reformulated as saying that *there is a natural Lorentz structure* $\tilde{g}$ *on the space* $\tilde{\boldsymbol{H}} := \boldsymbol{L}^{-1} \otimes \boldsymbol{H}$, namely

$$\tilde{g}(\boldsymbol{\phi}_1 \otimes v_1, \boldsymbol{\phi}_2 \otimes v_2) := (\boldsymbol{\phi}_1 \otimes \boldsymbol{\phi}_2)(v_1, v_2) := g_{\boldsymbol{\phi}_1 \otimes \boldsymbol{\phi}_2}(v_1, v_2) ,$$

(since $\boldsymbol{\phi}_1 \otimes \boldsymbol{\phi}_2 \in \boldsymbol{L}^{-2}$). Note also that $\tilde{\boldsymbol{\Lambda}}^2 = \boldsymbol{L}^{-1} \otimes \boldsymbol{\Lambda}^2$ and its square root $\tilde{\boldsymbol{\Lambda}} := \boldsymbol{L}^{-1/2} \otimes \boldsymbol{\Lambda}$ are Hermitian 1-dimensional spaces.

Moreover we set $\boldsymbol{U} := \boldsymbol{L}^{-1/2} \otimes \boldsymbol{S}$, so that $\tilde{\boldsymbol{H}}$ is the Hermitian subspace of $\boldsymbol{U} \otimes \overline{\boldsymbol{U}}$.

A basis $(\tilde{\zeta}_A := |\varepsilon|^{1/2} \otimes \zeta_A)$ of $\boldsymbol{U}$ turns out to be normalized, namely $2\, \tilde{\zeta}_1 \wedge \tilde{\zeta}_2$ is normalized with respect to the above said Hermitian structure of $\tilde{\boldsymbol{\Lambda}}^2$. A general normalized element $\omega \in \tilde{\boldsymbol{\Lambda}}^{-2}$ has the expression $\omega = e^{it}\, |\varepsilon|^{-1} \otimes \varepsilon$, $t \in \mathrm{R}$. Its inverse is $\omega^{-1} = e^{-it}\, |\varepsilon| \otimes \varepsilon^{-it}$.

We introduce the basis $(\tilde{\tau}_\lambda)$ of $\tilde{\boldsymbol{H}} := \boldsymbol{L}^{-1} \otimes \boldsymbol{H}$ and its dual basis $(\tilde{t}^\lambda)$ of $\tilde{\boldsymbol{H}}^* = \boldsymbol{L} \otimes \boldsymbol{H}^*$ given by

$$\tilde{\tau}_\lambda := \tfrac{1}{\sqrt{2}}\, |\varepsilon| \otimes \sigma_\lambda = |\varepsilon| \otimes \tau_\lambda \quad , \qquad \tilde{t}^\lambda := \sqrt{2}\, |\varepsilon|^{-1} \otimes s^\lambda = |\varepsilon|^{-1} \otimes t^\lambda .$$

These turn out to be orthonormal bases, namely

$$\tilde{g} = \eta_{\lambda\mu}\, \tilde{t}^\lambda \otimes \tilde{t}^\mu \quad , \qquad \tilde{g}^{-1} = \eta^{\lambda\mu}\, \tilde{\tau}_\lambda \otimes \tilde{\tau}_\mu .$$

The volume form $\tilde{\eta}$ on $\tilde{\boldsymbol{H}}$ determined (up to sign) by $\tilde{g}$ has the coordinate expression

$$\tilde{\eta} = \pm \tfrac{1}{4!}\, \varepsilon_{\lambda\mu\nu\rho}\, \tilde{t}^\lambda \wedge \tilde{t}^\mu \wedge \tilde{t}^\nu \wedge \tilde{t}^\rho = \pm \tilde{t}^0 \wedge \tilde{t}^1 \wedge \tilde{t}^2 \wedge \tilde{t}^3 .$$

Since an element of $\boldsymbol{L}^{-1} \otimes \boldsymbol{S} \otimes \overline{\boldsymbol{S}} = \boldsymbol{U} \otimes \overline{\boldsymbol{U}}$ can be seen as a linear map $\overline{\boldsymbol{U}}^{\mathrm{F}} \to \boldsymbol{U}$, while an element of $\boldsymbol{L} \otimes \overline{\boldsymbol{S}}^{\mathrm{F}} \otimes \boldsymbol{S}^{\mathrm{F}} = \overline{\boldsymbol{U}}^{\mathrm{F}} \otimes \boldsymbol{U}^{\mathrm{F}}$ can be seen as a linear map $\boldsymbol{U} \to \overline{\boldsymbol{U}}^{\mathrm{F}}$, we have a natural Clifford map

$$\tilde{\gamma} := \sqrt{2}\Big(\mathbf{1} + (\tilde{\mathbf{g}}^\flat)^{\mathrm{T}}\Big) : \tilde{\boldsymbol{H}} \to \mathrm{End}(\tilde{\boldsymbol{W}}) ,$$

where[3]

$$\tilde{\boldsymbol{W}} := \boldsymbol{U} \oplus \overline{\boldsymbol{U}}^{\mathrm{F}} = (\boldsymbol{L}^{-1/2} \otimes \boldsymbol{S}) \oplus (\boldsymbol{L}^{1/2} \otimes \overline{\boldsymbol{S}}^{\mathrm{F}}) .$$

Its basis expression is

$$\tilde{\gamma} = \sqrt{2}\, (\tilde{\tau}_\lambda + \eta_{\lambda\mu} \bar{\tilde{t}}^\mu) \otimes \tilde{t}^\lambda = \sqrt{2}\, (\tau_\lambda + \ \eta_{\lambda\mu} |\varepsilon|^{-2} \otimes \bar{t}^\mu) \otimes t^\lambda .$$

We also consider the 'contravariant' Clifford map

$$\tilde{\gamma}^\# := \tilde{\gamma} \circ \tilde{g}^\# : \tilde{\boldsymbol{H}}^* \to \mathrm{End}(\tilde{\boldsymbol{W}}) ,$$

with basis expression

$$\tilde{\gamma}^\# = \sqrt{2}\, (\eta^{\lambda\mu}\, \tilde{\tau}_\mu + \bar{\tilde{t}}^\lambda) \otimes \tilde{\tau}_\lambda = \sqrt{2}\, (\eta^{\lambda\mu}\, |\varepsilon|^2\, \tau_\mu + \bar{t}^\lambda) \otimes \tau_\lambda .$$

Given $\psi = u + \alpha \in \tilde{\boldsymbol{W}} := \boldsymbol{U} \oplus \overline{\boldsymbol{U}}^{\mathrm{F}}$ then its 'Dirac adjoint' is $\bar{\psi} = \bar{\alpha} + \bar{u} \in \boldsymbol{U}^{\mathrm{F}} \oplus \overline{\boldsymbol{U}} \equiv \tilde{\boldsymbol{W}}^{\mathrm{F}}$. Similarly to §2.3 we find a distinguished time-orientation of $\tilde{\boldsymbol{H}}$.

A normalized element $\omega \in \tilde{\boldsymbol{\Lambda}}^{-2}$ yields the antilinear map $\mathcal{C}_\omega : \tilde{\boldsymbol{W}} \to \tilde{\boldsymbol{W}}$ given by

$$\mathcal{C}_\omega(\sqcap + \alpha) := \omega^\#(\alpha) - \bar{\omega}^\flat(\sqcap) = \rceil^{-)\sqcup} \left(|\varepsilon| \otimes \varepsilon^\#(\alpha) - |\varepsilon|^{-\infty} \otimes \bar{\varepsilon}^\flat(\sqcap)\right) .$$

We have $\mathcal{C}_\omega \circ \mathcal{C}_\omega = \mathbf{1}_{\tilde{\boldsymbol{W}}}$. Then $\mathcal{C}_\omega$ is the *charge conjugation* associated with ω.

[3] More generally we can take $\tilde{\boldsymbol{W}} := \boldsymbol{L}^r \otimes (\boldsymbol{U} \oplus \overline{\boldsymbol{U}}^{\mathrm{F}})$, with r any rational number.

2.5 Bases transformations

In order to exploit the gauge symmetries of our formulation we deal with bases transformations induced by a general base transformation of $\boldsymbol{S}$. The set of all bases of $\boldsymbol{S}$ is group-affine with 'derived' group $Gl(2,\mathbb{C})$, namely if we fix a basis (ζ_A) then any other basis is given by

$$\zeta'_A = K^B_A \zeta_B \,, \quad (K^B_A) \in Gl(2,\mathbb{C}) \,.$$

Then we write the induced bases transformations. We obtain

$$\begin{aligned}
\varepsilon' &= \det K^{-1}\,\varepsilon \,, \\
|\varepsilon'| &= |\det K|^{-1}\,|\varepsilon| \,, \\
\sigma'_\lambda &= (\sigma_\lambda{}^{AB^{\cdot}}\,K^C_A\,\bar{K}^{D^{\cdot}}_{B^{\cdot}}\,s^\mu{}_{CD^{\cdot}})\,\sigma_\mu \,, \\
\tilde{\tau}'_\lambda &= |\det K|^{-1}\,(\sigma_\lambda{}^{AB^{\cdot}}\,K^C_A\,\bar{K}^{D^{\cdot}}_{B^{\cdot}}\,s^\mu{}_{CD^{\cdot}})\,\tilde{\tau}_\mu \,.
\end{aligned}$$

Thus, indicating by $K_{\boldsymbol{X}}$ the transformation matrix induced on the space $\boldsymbol{X}$ we have

$$\begin{aligned}
K_{\boldsymbol{\Lambda}^2} &= \det K \,, \\
K_{\boldsymbol{L}} &= |\det K| \,, \\
K_{\hat{\boldsymbol{\Lambda}}^2} &= |\det K|^{-1}\det K = \exp(i\arg\det K) \,, \\
(K_{\boldsymbol{H}})^\mu_\lambda &= \sigma_\lambda{}^{AB^{\cdot}}\,K^C_A\,\bar{K}^{D^{\cdot}}_{B^{\cdot}}\,s^\mu{}_{CD^{\cdot}} \,, \\
\tilde{K}^\mu_\lambda := (K_{\tilde{\boldsymbol{H}}})^\mu_\lambda &= |\det K|^{-1}\,(K_{\boldsymbol{H}})^\mu_\lambda \,, \\
K_{\boldsymbol{U}} &= |\det K|^{-1/2}\,K \,.
\end{aligned}$$

Since $Gl(2,\mathbb{C})$ is connected we see that $K_{\boldsymbol{H}}$ and $K_{\tilde{\boldsymbol{H}}}$ preserve orientation, namely we have a way of selecting 'positive' orientations on $\boldsymbol{H}$ and $\tilde{\boldsymbol{H}}$: those orientations for which the 'Pauli bases' (σ_λ) and $(\tilde{\tau}_\lambda)$ induced by any basis of $\boldsymbol{S}$ are positively oriented.

3 Two-spinor connections

We consider a 4-dimensional manifold $\boldsymbol{M}$ and a complex vector bundle $\boldsymbol{S} \to \boldsymbol{M}$ with 2-dimensional fibres. Linear fibred coordinates are indicated by (x^a, z^A). According to the constructions of the previous section, we now have several vector bundles over $\boldsymbol{M}$, with smooth natural structures. Moreover we have the semi-vector bundles $\boldsymbol{L}^r$, with r rational. But observe that, for given r, according to the bundle topology we may have no $\boldsymbol{\Lambda}^r = (\boldsymbol{\Lambda}^2)^{r/2}$, or we may have one or more (not isomorphic) such bundles. Though these will not be essential to our treatment, note that we obtain a unique $\boldsymbol{\Lambda}^r$ by a suitable restriction of the base manifold.

We shall consider a complex-linear connection B on $\boldsymbol{S} \to \boldsymbol{M}$, whose coefficients $\mathrm{B}_a{}^A{}_B : \boldsymbol{M} \to \mathbb{C}$ will be also denoted by

$$\mathrm{B}_a{}^A{}_B := \mathrm{B}^\lambda_a\,\sigma_\lambda{}^A{}_B \,, \tag{3.1}$$

with $\mathrm{B}^\lambda_a : \boldsymbol{M} \to \mathbb{C}$.

3.1 Two-spinor connections and Lorentz structure

We shall be involved with the connections on $\mathbf{\Lambda}^2$ and $\boldsymbol{L}$ induced by B. In particular we have

$$\nabla\varepsilon = 2\,(G_a + i\,A_a)\,dx^a \otimes \varepsilon\ ,$$
$$\nabla|\varepsilon| = 2\,G_a\,dx^a \otimes |\varepsilon|\ .$$

with $G_a\,, A_a : \boldsymbol{M} \to \mathrm{R}$ given by

$$A_a = \tfrac{1}{4i}(\mathrm{B}_{a}{}^{A}{}_{A} - \bar{\mathrm{B}}_{a}{}^{A^{\cdot}}{}_{A^{\cdot}}) = \tfrac{1}{2i}(\mathrm{B}^0_a - \bar{\mathrm{B}}^0_a)\ ,$$
$$G_a = \tfrac{1}{4}(\mathrm{B}_{a}{}^{A}{}_{A} + \bar{\mathrm{B}}_{a}{}^{A^{\cdot}}{}_{A^{\cdot}}) = \tfrac{1}{2}(\mathrm{B}^0_a + \bar{\mathrm{B}}^0_a)\ .$$

Moreover,

$$\nabla\boldsymbol{\phi} = (d\log\phi + 2G) \otimes \boldsymbol{\phi}\ .$$

By straightforward calculation, recalling (3.1) and the definition of the coefficients $\Sigma_{\lambda\mu\nu}$ (§2.2), one finds

Proposition 3.1 *Let* B *be a complex-linear connection on* $\boldsymbol{S} \to \boldsymbol{M}$*. The coefficients* $\mathrm{B}_{a}{}^{\mu}{}_{\lambda}$ *of the connection induced on* $\boldsymbol{S} \otimes \overline{\boldsymbol{S}}$*, in the basis* (τ_λ)*, are given by*

$$\mathrm{B}_{a}{}^{\mu}{}_{\lambda} = 2(\mathrm{B}^\nu_a\,\Sigma_{\nu}{}^{\mu}{}_{\lambda} + \bar{\mathrm{B}}^\nu_a\,\bar{\Sigma}_{\nu}{}^{\mu}{}_{\lambda})\ .$$

Its coefficients being real, this connection is reducible to $\boldsymbol{H}$*.*

Next we consider the connection $\tilde{\Gamma}$ induced on $\tilde{\boldsymbol{H}} := \boldsymbol{L}^{-1} \otimes \boldsymbol{H}$. Its coefficients in the basis $(\tilde{\tau}_\lambda)$ turn out to be

$$\tilde{\Gamma}_{a}{}^{\mu}{}_{\lambda} = \mathrm{B}_{a}{}^{\mu}{}_{\lambda} - 2\,G_a\,\delta^{\mu}{}_{\lambda}\ .$$

Again by straightforward calculation one finds that $\tilde{\Gamma}$ is metric, that is $\nabla[\tilde{\Gamma}]\tilde{g} = 0$ (hence the coefficients $\tilde{\Gamma}_{a}{}^{\mu}{}_{\lambda}$ are antisymmetric).

Proposition 3.2 *We have*

$$\mathrm{B}_{a}{}^{A}{}_{C} = (G_a + i\,A_a)\delta^{A}{}_{C} + \tfrac{1}{4}\,\tilde{\Gamma}_{a}{}^{\lambda}{}_{\mu}\,\Sigma_{\lambda}{}^{\mu\nu}\sigma_{\nu}{}^{C}{}_{A}\ .$$

So, B is completely determined by $\tilde{\Gamma}$ and by the connection induced on $\mathbf{\Lambda}^2$ (locally, $\mathbf{\Lambda}$); the latter, in turn, splits into its real part G (connection on $\boldsymbol{L}$) and its immaginary part A (locally, the connection on the Hermitian line bundle $\tilde{\mathbf{\Lambda}}$; later on, this term will be interpreted as the electromagnetic potential).

The formula of proposition 3.1 can be expressed, by separating the 'timelike' index 0 from 'spacelike' indices (indicated by latin letters $p, q, r, ..$), as

$$G_a = \tfrac{1}{2}\mathrm{B}_{a}{}^{0}{}_{0} = \tfrac{1}{2}(\mathrm{B}^0_a + \bar{\mathrm{B}}^0_a)\ ,$$
$$\tilde{\Gamma}_{a}{}^{r}{}_{q} = (\mathrm{B}^0_a + \bar{\mathrm{B}}^0_a)\delta^r_q - i(\mathrm{B}^p_a - \bar{\mathrm{B}}^p_a)\varepsilon^{r}{}_{qp}\ ,$$
$$\tilde{\Gamma}_{a}{}^{\mu}{}_{0} = \mathrm{B}^\mu_a + \bar{\mathrm{B}}^\mu_a\ ,$$
$$\tilde{\Gamma}_{a}{}^{0}{}_{q} = (\mathrm{B}^p_a + \bar{\mathrm{B}}^p_a)\delta_{pq}\ ,$$

with $\mathrm{B}^\lambda_a = \mathrm{B}_{a}{}^{A}{}_{B}\,\hat{s}^{\lambda}{}_{A}{}^{B}$, namely $\mathrm{B}^0_a = G_a + i\,A_a\,,\ \mathrm{B}^p_a = \frac{1}{4}\,\tilde{\Gamma}_{a}{}^{\lambda}{}_{\mu}\,\Sigma_{\lambda}{}^{\mu p}$.

Similarly, proposition 3.2 can be expressed as

$$\mathrm{B}_{a}{}^{A}{}_{B} = (G_a + i\,A_a)\,\delta^{A}{}_{B} + (\tfrac{i}{4}\,\varepsilon_{r}{}^{sp}\,\tilde{\Gamma}_{a}{}^{r}{}_{s} + \tfrac{1}{2}\,\tilde{\Gamma}_{a}{}^{p}{}_{0})\,\sigma_{p}{}^{A}{}_{B}\ .$$

A further equivalent expression of the above formulas can be written in 4-spinor formalism [CJ96] as

$$\tilde{\Gamma}_{a}{}^{\mu}{}_{\lambda} = g^{\mu\nu}\,\mathrm{B}_{a}{}^{A}{}_{B}\,(\gamma_\nu \wedge \gamma_\lambda)^{B}{}_{A}\ ,$$

and

$$\mathrm{B}_{a}{}^{A}{}_{B} = (G_a + iA_a)\,\delta^{A}{}_{B} + \tfrac{1}{4}\tilde{\Gamma}_{a}{}^{\lambda\mu}(\gamma_\lambda \wedge \gamma_\mu)^{A}{}_{B}\ ,$$

with $A, B = 1, .., 4$.

3.2 Two-spinor connections and gauge transformations

We are interested in seeing how the coefficients of the involved connections change under a *gauge transformation*, namely a linear transformation of the fibre coordinates.

Let $\zeta'_A = K^B_A\,\zeta_B$. We obtain

$$\mathrm{B}'^{\,A}_{a\ B} = (K^{-1})^A_C\,K^D_B\,\mathrm{B}_a{}^C{}_D - (K^{-1})^A_C\,\partial_a K^C_B\ ,$$

$$G'_a = G_a - \partial_a \log|\det K|\ ,$$

$$A'_a = A_a - \partial_a \arg\det K\ ,$$

$$\tilde\Gamma'^{\,\lambda}_{a\ \mu} = (\tilde K^{-1})^\lambda_\nu\,\tilde K^\rho_\mu\,\tilde\Gamma_a{}^\nu{}_\rho - (\tilde K^{-1})^\lambda_\nu\,\partial_a\tilde K^\nu_\mu\ .$$

3.3 Two-spinor curvature

If $\boldsymbol{X}\to\boldsymbol{M}$ is any one of the bundles derived from $\boldsymbol{S}$, then we indicate by $R_{\boldsymbol{X}}:\boldsymbol{M}\to\wedge^2T^*\boldsymbol{M}\otimes\mathrm{End}(\boldsymbol{X})$ the curvature tensor of the connection induced by B on it. First, we can easily calculate:

$$\begin{aligned}R_{\boldsymbol{L}} &= -2\,dG\otimes\mathbf{1}_{\boldsymbol{L}}\ ,\\ R_{\boldsymbol{\Lambda}^2} &= -2\,(dG+i\,dA)\otimes\mathbf{1}_{\boldsymbol{\Lambda}^2}\ ,\\ R_{\tilde{\boldsymbol{\Lambda}}^2} &= -2i\,dA\otimes\mathbf{1}_{\tilde{\boldsymbol{\Lambda}}^2}\ ,\end{aligned}$$

where $\mathbf{1}_{\boldsymbol{X}}:\boldsymbol{M}\to\mathrm{End}(\boldsymbol{X})$ denotes the identity endomorphism of $\boldsymbol{X}$ over $\boldsymbol{M}$.

Next we see how the curvature tensor of a two-spinor connection B can be expressed in terms of the curvature tensors of the connections induced on $\tilde{\boldsymbol{H}}$, $\boldsymbol{L}$ and $\tilde{\boldsymbol{\Lambda}}^2$ (see also [GP82]).

Similarly to (3.1), we indicate by $P_{ab}{}^A{}_B := (R_{\boldsymbol{S}})_{ab}{}^A{}_B$, $(R_{\boldsymbol{H}})_{ab}{}^\mu{}_\lambda$ and $\tilde R_{ab}{}^\mu{}_\lambda := (R_{\tilde{\boldsymbol{H}}})_{ab}{}^\mu{}_\lambda$ the components of the curvature tensors of B, $\mathrm{B}_{\boldsymbol{H}}$ and $\tilde\Gamma := \mathrm{B}_{\tilde{\boldsymbol{H}}}$ in the frames (ζ_A), (σ_λ) and $(\tilde\tau_\lambda)$, respectively. Moreover we set

$$P_{ab}{}^A{}_B := P^\lambda_{ab}\,\sigma_\lambda{}^A{}_B\ .$$

Then we obtain

$$\begin{aligned}(R_{\boldsymbol{H}})_{ab}{}^\mu{}_\lambda &= 2(P^\nu_{ab}\,\Sigma_\nu{}^\mu{}_\lambda + \bar P^\nu_{ab}\,\bar\Sigma_\nu{}^\mu{}_\lambda)\ ,\\ \tilde R_a{}^\mu{}_\lambda &= P_{ab}{}^\mu{}_\lambda - 2\,(dG)_{ab}\,\delta^\mu{}_\lambda\ ,\\ P_{ab}{}^A{}_B &= -(dG+i\,dA)_{ab}\,\delta^A{}_B + \tfrac14\,\tilde R_a{}^\lambda{}_\mu\,\Sigma_\lambda{}^{\mu\nu}\sigma_\nu{}^C{}_A\ .\end{aligned}$$

Again we can re-express our results in terms of 'timelike' index from 'spacelike' indices. In particular

$$P_{ab}{}^A{}_B = -(dG+i\,dA)_{ab}\,\delta^A{}_B + (\tfrac{i}{4}\,\varepsilon_r{}^{sp}\,\tilde R_{ab}{}^r{}_s + \tfrac12\,\tilde R_{ab}{}^p{}_0)\,\sigma_p{}^A{}_B\ .$$

Moreover we have the equivalent 4-spinor expressions

$$\tilde R_{ab}{}^\mu{}_\lambda = g^{\mu\nu}\,P_{ab}{}^A{}_B\,(\gamma_\nu\wedge\gamma_\lambda)^B{}_A\ ,$$

and

$$P_{ab}{}^A{}_B = -(dG+i\,dA)_{ab}\,\delta^A{}_B + \tfrac14\tilde R_{ab}{}^{\lambda\mu}(\gamma_\lambda\wedge\gamma_\mu)^A{}_B\ ,$$

with $A,B=1,\dots,4$.

3.4 Unit space of lengths

An important simplification arises when the curvature tensor of the connection induced on $\boldsymbol{L}$ vanishes, i.e. $dG = 0$. In that case we can 'gauge away' the conformal part of the connection, namely we can take local bundle charts such that $G = 0$, namely local frames (ζ_A) of $\boldsymbol{S}$ such that $\nabla|\varepsilon| = 0$ (§2.2). Moreover we now have the distinguished set of constant local fields $\boldsymbol{M} \to \boldsymbol{L}$, so that we can regard the bundle $\boldsymbol{L}$, at least locally, as a vector space, which we identify with the *unit space of lengths* [CJ96, CJM95].

In other terms, the conformal structure of $\boldsymbol{H}$ can be now reduced to constant conformal factors by taking only those fields $\boldsymbol{\phi} : \boldsymbol{M} \to \boldsymbol{L}^{-1}$ such that $\nabla\boldsymbol{\phi} = 0$.

4 Soldering form (vierbein)

4.1 Algebraic properties

We consider a linear morphism $\theta : T\boldsymbol{M} \to \boldsymbol{H}$, i.e. a section

$$\theta : \boldsymbol{M} \to T^*\boldsymbol{M} \otimes \boldsymbol{H}$$

(all tensor products are over $\boldsymbol{M}$). Its coordinate expression is

$$\theta = \theta^\lambda_a\, \tau_\lambda \otimes dx^a\ ,$$

with $\theta^\lambda_a : \boldsymbol{M} \to \mathrm{R}$. Note that we can write $\theta : \boldsymbol{L}^* \otimes T\boldsymbol{M} \to \tilde{\boldsymbol{H}}$, so that we can view θ as a 'scaled' *soldering form*, or *vierbein*. In general we wish to allow for the case when θ is degenerate, namely not an isomorphism.

From θ we obtain the following objects

$$\begin{aligned} g &:= \theta^*\tilde{g} : \boldsymbol{M} \to \boldsymbol{L}^2 \otimes T^*\boldsymbol{M} \otimes T^*\boldsymbol{M}\ , \\ \eta &:= \theta^*\tilde{\eta} = \tilde{\eta} \,\lrcorner\, \wedge^4\theta : \boldsymbol{M} \to \boldsymbol{L}^4 \otimes \wedge^4 T^*\boldsymbol{M}\ , \\ \gamma &:= \tilde{\gamma} \circ \theta : T\boldsymbol{M} \to \boldsymbol{L} \otimes \mathrm{End}(\tilde{\boldsymbol{W}})\ , \end{aligned}$$

with $\tilde{\boldsymbol{W}} = \boldsymbol{U} \oplus \overline{\boldsymbol{U}}^{\mathrm{F}} := (\boldsymbol{L}^{-1/2} \times \boldsymbol{S}) \oplus (\boldsymbol{L}^{1/2} \otimes \overline{\boldsymbol{S}}^{\mathrm{F}})$. We have the following coordinate expressions:

$$g = \eta_{\lambda\mu}\, \theta^\lambda_a\, \theta^\mu_b\, |\varepsilon|^{-2} \otimes dx^a \otimes dx^b\ ,$$

$$\begin{aligned} \eta &= \tfrac{1}{4!}\, \varepsilon_{\lambda\mu\nu\rho}\, \theta^\lambda_a\, \theta^\mu_b\, \theta^\nu_c\, \theta^\rho_d\, |\varepsilon|^{-4} \otimes dx^a \wedge dx^b \wedge dx^c \wedge dx^d = \\ &= \tfrac{1}{4!}\, \varepsilon_{\lambda\mu\nu\rho}\, \varepsilon^{abcd}\, \theta^\lambda_a\, \theta^\mu_b\, \theta^\nu_c\, \theta^\rho_d\, |\varepsilon|^{-4} \otimes \xi = \\ &= \det(\theta)\, |\varepsilon|^{-4} \otimes \xi\ , \end{aligned}$$

$$\gamma = \sqrt{2}\, \theta^\lambda_a\, |\varepsilon|^{-1} \otimes (\tau_\lambda + \eta_{\lambda\mu}\bar{t}^\mu) \otimes dx^a\ ,$$

where $\xi := dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$.

If and only if θ is non-degenerate, the above objects turn out to be a conformal Lorentz metric, the corresponding (up to sign, unique) conformal volume form and a conformal Clifford map.

The frame (τ_λ) of $\boldsymbol{H}$ together with θ yields the four 1-forms

$$\theta^\lambda := \theta^* t^\lambda = \theta^\lambda_a\, dx^a\ .$$

Iff θ is non-degenerate these constitute a frame of $T^*\boldsymbol{M}$, the dual frame of

$$\theta_\lambda := \theta^{-1}(\tau_\lambda) = \theta^a_\lambda\, \partial_a\ ,$$

with $\theta^a_\lambda := (\theta^{-1})^a_\lambda$.

4.2 Vierbein and connection

Consider a two-spinor connection B and the induced connection $\tilde{\Gamma}$ on $\tilde{\boldsymbol{H}}$. Let moreover θ be a non-degenerate vierbein and Γ a linear connection on $\boldsymbol{TM}$. We denote by $\Gamma_{a\,c}^{\,b}$ and $\Gamma_{a\,\mu}^{\,\lambda}$ the coefficients of Γ in the frames (∂_a) and (θ_λ), respectively, and by $\tilde{\Gamma}_{a\,\mu}^{\,\lambda}$ the coefficients of $\tilde{\Gamma}$ in the frame (τ_λ). Then the condition $\nabla\theta = 0$ can be expressed in coordinates by either of the two equivalent formulas:

$$\tilde{\Gamma}_{a\,\mu}^{\,\lambda} + 2\,G_a\,\delta^{\lambda}_{\ \mu} = \Gamma_{a\,\mu}^{\,\lambda}\,, \tag{4.1}$$

$$\partial_a\theta^\lambda_b - (\tilde{\Gamma}_{a\,\mu}^{\,\lambda} + 2\,G_a\,\delta^{\lambda}_{\ \mu})\,\theta^\mu_b + \Gamma_{a\,b}^{\,c}\,\theta^\lambda_c = 0\,. \tag{4.2}$$

Then we obtain:

Proposition 4.1 *Let* B *be a two-spinor connection and* θ *a non-degenerate vierbein. Then there exists a unique connection* Γ *on* $\boldsymbol{TM} \to \boldsymbol{M}$ *such that* $\nabla[\Gamma\otimes\tilde{\Gamma}]\theta = 0$.

Moreover we have $\nabla[\Gamma]g = 0$.

We recall that the *Frölicher-Nijenhuis bracket* of $\mathrm{B}_{\boldsymbol{H}}$ seen as a tangent-valued 1-form

$$\mathrm{B}_{\boldsymbol{H}} : \boldsymbol{H} \to T^*\boldsymbol{M} \underset{\boldsymbol{H}}{\otimes} T\boldsymbol{H}$$

and of $\theta : \boldsymbol{M} \to T^*\boldsymbol{M}\otimes\boldsymbol{H}$ seen as a tangent-valued 1-form

$$\theta : \boldsymbol{H} \to T^*\boldsymbol{M} \underset{\boldsymbol{H}}{\otimes} V\boldsymbol{H}$$

is a tangent-valued 2-form [M91]

$$[\mathrm{B}_{\boldsymbol{H}}\,,\theta] : \boldsymbol{H} \to \wedge^2 T^*\boldsymbol{M} \underset{\boldsymbol{H}}{\otimes} V\boldsymbol{H}\ ,$$

with coordinate expression

$$\begin{aligned}[\mathrm{B}_{\boldsymbol{H}}\,,\theta]_{ab}{}^{\lambda} &= \partial_a\theta^\lambda_b - \partial_b\theta^\lambda_a - \mathrm{B}_{a\,\mu}^{\,\lambda}\,\theta^\mu_b + \mathrm{B}_{b\,\mu}^{\,\lambda}\,\theta^\mu_a = \\ &= \partial_a\theta^\lambda_b - \partial_b\theta^\lambda_a - \tilde{\Gamma}_{a\,\mu}^{\,\lambda}\,\theta^\mu_b + \tilde{\Gamma}_{b\,\mu}^{\,\lambda}\,\theta^\mu_a - 2\,G_a\,\theta^\lambda_b + 2\,G_b\,\theta^\lambda_a\ .\end{aligned}$$

Moreover, note that $[\mathrm{B}_{\boldsymbol{H}}\,,\theta]$ can be seen as a section

$$[\mathrm{B}_{\boldsymbol{H}}\,,\theta] : \boldsymbol{M} \to \wedge^2 T^*\boldsymbol{M}\otimes\boldsymbol{H}\ .$$

Proposition 4.2 *The torsion of the connection* Γ *of proposition 4.1 is given by*

$$T[\Gamma]\,\mathrm{y}\,\theta = [\mathrm{B}_{\boldsymbol{H}}\,,\theta]\ ,$$

or, in coordinates, $T[\Gamma] = T_{ab}{}^{c}\,dx^b\wedge dx^c\otimes\partial_c$ *with*

$$T^a{}_{bc}\,\theta^\lambda_a = [\mathrm{B}_{\boldsymbol{H}}\,,\theta]_{ab}{}^{\lambda}\ .$$

By the way we note that $\mathrm{B}'_{\boldsymbol{H}} := \mathrm{B}_{\boldsymbol{H}} + \theta$ is a 'polynomial' connection [MM91] of degree 1. Conversely, any polynomial connection of degree 1 on $\boldsymbol{H}$ this splits canonically in the linear part $\mathrm{B}_{\boldsymbol{H}}$ and a vierbein θ. Moreover, $\mathrm{B}'_{\boldsymbol{H}}$ is a gauge connection of the conformal Poincaré group.

If $dG = 0$ (§3.4), so that the conformal structure of $\boldsymbol{H}$ can be reduced to constant conformal factors, then also the conformal structure of $\boldsymbol{TM}$ induced by a non-degenerate θ is accordingly reduced to constant conformal factors (following [CJM95, CJ96] we call this a 'scaled' Lorentz structure). From $\nabla[\Gamma]g = 0$ (proposition 4.1), we have that Γ is now a metric connection in the usual sense. Gauging away the field G corresponds to taking frames of $\boldsymbol{H}$ and $\boldsymbol{TM}$ with constant scalar products $g_{\lambda\mu} = \phi^2\,\eta_{\lambda\mu}\,,\ \nabla\phi = 0$. Furthermore, besides Γ we also have on $\boldsymbol{TM}$ the metric torsionless connection induced by g. The difference between these two connections is just the torsion of Γ, which now reads

$$T^a{}_{bc}\,\theta^\lambda_a = [\tilde{\Gamma}\,,\theta]_{ab}{}^{\lambda} = \partial_a\theta^\lambda_b - \partial_b\theta^\lambda_a - \tilde{\Gamma}_{a\,\mu}^{\,\lambda}\,\theta^\mu_b + \tilde{\Gamma}_{b\,\mu}^{\,\lambda}\,\theta^\mu_a\ .$$

5 Fields and field equations

5.1 The fields

We are going to formulate a Lagrangian theory of Einstein-Cartan-Maxwell-Dirac fields according to our two-spinor approach. The metric will be essentially represented by a vierbein θ, which can be seen as a kind of 'square root' of g; the spacetime connection is represented by the traceless part of the two-spinor connection B, while the electromagnetic potential is represented by the connection A induced, possibly locally, on $\tilde{\Lambda}$ (see also [CJ96, GP75, IW33]). The relation between the spacetime connection and the metric will result from the field equations as in the standard 'metric-affine' approach [FK83, GH96, HCMN95, R83]). The equation for the field G will be $dG = 0$, so that the observations of §3.4 and at the end of §4.2 apply: the Lorentz structure induced by θ is scaled by a *global* (constant) conformal factor. Moreover we shall not consider non-constant 'dilaton' fields (see for example [HCMN95]). More simply, but equivalently in the present context, we could just restrict us to considering spin connections such that the curvature tensor vanishes on $\boldsymbol{L}$.

The Dirac spinor field is represented by a section

$$\psi = u + \alpha : \boldsymbol{M} \to \boldsymbol{W} := \boldsymbol{L}^{-3/2} \otimes \boldsymbol{U} \oplus \overline{\boldsymbol{U}}^{\mathrm{F}} := (\boldsymbol{L}^{-2} \otimes \boldsymbol{S}) \oplus (\boldsymbol{L}^{-1} \otimes \overline{\boldsymbol{S}}^{\mathrm{F}}) \ .$$

Because of the factor $\boldsymbol{L}^{-3/2}$ we have $\langle \bar{\psi}, \psi \rangle : \boldsymbol{M} \to \boldsymbol{L}^{-3}$; this will yield correct units of measurement for various objects, in particular for current and probability density [CJ96].

Summarizing, the fields will be θ, B and $\psi = u + \alpha$, represented in components by

$$\theta_a^\lambda \ , \quad G_a \ , \quad A_a \ , \quad \Gamma_{a}{}^{\lambda}{}_{\mu} \ , \quad u^A \ , \quad \bar{u}^{A^\bullet} \ , \quad \alpha_{A^\bullet} \ , \quad \bar{\alpha}_A \ .$$

We shall take the usual Lagrangians for Einstein, Maxwell and Dirac fields and write down 'adapted' versions in terms of our fields. Morever we shall have a Lagrangian for the field G. Altogether we obtain a 'total' Lagrangian

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{\mathrm{g}} + \mathcal{L}_{\mathrm{em}} + \mathcal{L}_{\mathrm{D}} : J\boldsymbol{E} \to \wedge^4 T^* \boldsymbol{M} \ ,$$

where $\boldsymbol{E} \to \boldsymbol{M}$ is the bundle whose sections are the whole of our fields, and $J\boldsymbol{E}$ is its first-jet prolongation [MM83]. We shall also write $\mathcal{L} = \ell \xi$ with $\xi := dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ and

$$\ell = \ell_G + \ell_{\mathrm{g}} + \ell_{\mathrm{em}} + \ell_{\mathrm{D}} : J\boldsymbol{E} \to \mathrm{R} \ .$$

For each term in the Lagrangian we shall write down the corresponding contribution to the Euler-Lagrange operator. This is a map [MM83]

$$\mathcal{E} = \mathcal{E}_G + \mathcal{E}_{\mathrm{g}} + \mathcal{E}_{\mathrm{em}} + \mathcal{E}_{\mathrm{D}} : J\boldsymbol{E} \to \wedge^4 T^* \boldsymbol{M} \otimes \boldsymbol{E}^* \ ,$$

where $\boldsymbol{E}^*$ is the *real dual* of $\boldsymbol{E}$.

We shall try to formulate an approach independent of the non-degeneracy of θ, thus obtaining a more general theory (see also [J84]). This however will be accomplished only partially, due to an obstruction in the electromagnetic part.

5.2 The field G

The Lagrangian for the field G is assumed to be

$$\mathcal{L}_G = dG \wedge dG \ .$$

The corresponding contribution to the Euler-Lagrange operator is simply dG. We shall see that, among the other terms in the total Lagrangian, G appears only in the Dirac part;

however the corresponding contribution to $\mathcal{E}$ vanishes, so that the equation for G turns out to be exactly $dG = 0$. As it was already said in §3.4, this implies the existence of constant sections of $\boldsymbol{L}$. One may expect that coupling constants arise naturally as constant sections of $\boldsymbol{L}^r$ (r rational). Actually we shall have mass $m \in \boldsymbol{L}^{-1}$ and the gravitational coupling constant $\boldsymbol{\kappa} \in \boldsymbol{L}^2$.

However, note that we could generalise this setting by allowing some coupling of G with the other fields. In that case we would be forced to consider non-constant dilaton and coupling factors [HCMN95].

5.3 Gravitational Lagrangian

In standard General Relativity the Lagrangian of the gravitational field is assumed to be the scalar curvature times the volume form. In the general case when θ is not necessarily a soldering form, scalar curvature cannot be defined, but we can write down a 4-form $\mathcal{L}_{\mathrm{g}}$ from θ and the curvature tensor $\tilde{R}$ of $\tilde{\Gamma}$. Moreover $\mathcal{L}_{\mathrm{g}}$ will turn out to equal the usual gravitational Lagrangian in the non-degenerate case.

The construction is as follows. First we raise via $\tilde{g}$ one index of $\tilde{R}$, thus obtaining the field

$$\tilde{R}^{\#} : \boldsymbol{M} \to \wedge^2 T^*\boldsymbol{M} \otimes \wedge^2 \tilde{\boldsymbol{H}} \ .$$

Then we set

$$\mathcal{L}_{\mathrm{g}} := \boldsymbol{\kappa}^{-1}\, 4!\, \tilde{\eta}(\tilde{R}^{\#} \wedge \theta \wedge \theta) : \boldsymbol{M} \to \wedge^4 T^*\boldsymbol{M} \ ,$$

where $\boldsymbol{\kappa} : \boldsymbol{M} \to \boldsymbol{L}^2$, with local expression $\boldsymbol{\kappa} = \kappa\, |\varepsilon|^2$, is the *gravitational coupling factor* (actually a constant, §5.2). We obtain the coordinate expression

$$\mathcal{L}_{\mathrm{g}} = \tfrac{1}{2}\kappa^{-1}\, \varepsilon_{\lambda\mu\nu\rho}\, R_{ab}{}^{\lambda\mu}\, \theta^\nu_c\, \theta^\rho_d\, dx^a \wedge dx^b \wedge dx^c \wedge dx^d \ ,$$

namely

$$\ell_{\mathrm{g}} = \tfrac{1}{2}\, \kappa^{-1}\, \varepsilon_{\lambda\mu\nu\rho}\, \varepsilon^{abcd}\, R_{ab}{}^{\lambda\mu}\, \theta^\nu_c\, \theta^\rho_d \ .$$

We calculate the θ- and Γ-components of $\mathcal{E}_{\mathrm{g}}$ obtaining

$$(\mathcal{E}_{\mathrm{g}})^c_\nu = \frac{\partial}{\partial\theta^\nu_c}\ell_{\mathrm{g}} = \kappa^{-1}\, \varepsilon_{\lambda\mu\nu\rho}\, \varepsilon^{abcd}\, R_{ab}{}^{\lambda\mu}\, \theta^\rho_d \ ,$$

$$\begin{aligned}(\mathcal{E}_{\mathrm{g}})^{a}{}_{\alpha}{}^{\beta} &= \frac{\partial}{\partial\Gamma_{a}{}^{\alpha}{}_{\beta}}\ell_{\mathrm{g}} - \partial_b\theta^\nu_c\, \frac{\partial}{\theta^\nu_c}\, \frac{\partial}{\partial(\partial_b\Gamma_{a}{}^{\alpha}{}_{\beta})}\ell_{\mathrm{g}} = \\ &= \kappa^{-1}\, \varepsilon^{abcd}\, \varepsilon_{\lambda\mu\nu\rho}\Big[2\, \partial_b\theta^\nu_c\, \theta^\rho_d\, \delta^\lambda_\alpha\, \eta^{\mu\beta} + (\eta^{\mu\beta}\, \Gamma_{b}{}^{\lambda}{}_{\alpha} + \delta^\lambda_\alpha\, \Gamma_{b}{}^{\mu\beta})\Big] \ .\end{aligned}$$

The antisymmetry of $(\mathcal{E}_{\mathrm{g}})^{a}{}_{\alpha}{}^{\beta}$ in the indices α and β is apparent from

$$(\mathcal{E}_{\mathrm{g}})^{a}{}_{\alpha\gamma} := (\mathcal{E}_{\mathrm{g}})^{a}{}_{\alpha}{}^{\beta}\, \eta_{\beta\gamma} = \kappa^{-1}\, \varepsilon^{abcd}\, \varepsilon_{\lambda\mu\nu\rho}\, \Big[2\, \partial_b\theta^\nu_c\, \theta^\rho_d\, \delta^\lambda_\alpha\, \delta^\mu_\gamma + (\delta^\lambda_\alpha\, \Gamma_{b}{}^{\mu}{}_{\gamma} - \delta^\lambda_\gamma\, \Gamma_{b}{}^{\mu}{}_{\alpha})\Big] \ .$$

Using $\varepsilon_{\alpha\beta\lambda\mu}\, \varepsilon^{\alpha\beta\nu\rho} = 2(\delta^\lambda_\nu \delta^\mu_\rho - \delta^\lambda_\rho \delta^\mu_\nu)$ we find

$$\begin{aligned}(\mathcal{E}_{\mathrm{g}})^{a}{}_{\alpha\beta} &= 2\, \kappa^{-1}\, \varepsilon^{abcd}\, \varepsilon_{\alpha\beta\nu\rho}\, (\partial_b\theta^\nu_c\, \theta^\rho_d - \Gamma_{b}{}^{\nu}{}_{\sigma}\, \theta^\sigma_c\, \theta^\rho_d) = \\ &= 2\, \kappa^{-1}\, \varepsilon^{abcd}\, \varepsilon_{\alpha\beta\nu\rho}\, [\tilde{\Gamma}, \theta]^\nu_{bc}\, \theta^\rho_d \ .\end{aligned}$$

So the Γ-component of the Euler-Lagrange operator gives, in the non-degenerate case, the torsion of the connection Γ induced by θ and $\tilde{\Gamma}$ on $T\boldsymbol{M}$.

Next we show that the θ-component of $\mathcal{E}_{\mathrm{g}}$ gives, in the non-degenerate case, the Ricci tensor. Actually let θ be an isomorphism, and take *any* given point $p_0 \in \boldsymbol{M}$. We can find base coordinates (x^a) such that $\theta^\rho_d(p_0) = \delta^\rho_d$, so that *at* p_0 we have

$$(\mathcal{E}_{\mathrm{g}})^c_\nu = 2\, \kappa^{-1}\, \delta^a_{[\lambda}\, \delta^b_\mu\, \delta^c_{\nu]}\, R_{ab}{}^{\lambda\mu} = 8\, \kappa^{-1}\, E_{\nu\mu}\, \eta^{\mu c} \ ,$$

where E is the Einstein tensor, $E_{\nu\mu} = R_{\nu\rho}{}^{\rho}{}_{\mu} - \frac{1}{2}\, \eta^{\alpha\beta}\, R_{\alpha\rho}{}^{\rho}{}_{\beta}\, \eta_{\nu\mu}$.

5.4 Dirac Lagrangian

The usual Dirac Lagrangian on curved spacetime [IZ80, CJ96] is

$$\mathcal{L}_{\mathrm{D}} = \Big[\tfrac{i}{2}\Big(\langle\bar\psi, \not\nabla\psi\rangle - \langle\overline{\not\nabla\psi}, \psi\rangle\Big) - \tfrac{mc}{\tilde{}}\,\langle\bar\psi,\psi\rangle\Big]\,\eta\;,$$

where $\not\nabla = \gamma^a\nabla_a$. In our two-spinor formalism

$$\psi = u + \alpha \in \boldsymbol{W} = \boldsymbol{L}^{-3/2}\otimes(\boldsymbol{U}\times\overline{\boldsymbol{U}}^{\mathrm{F}})\;,$$
$$\bar\psi = \bar\alpha + \bar u \in \boldsymbol{L}^{-3/2}\otimes(\boldsymbol{U}^{\mathrm{F}}\times\overline{\boldsymbol{U}}) = \boldsymbol{L}^{-3}\otimes\boldsymbol{W}^{\mathrm{F}}\;,$$

namely if $\psi' = v + \beta \in \boldsymbol{W}$ then

$$\langle\bar\psi, \psi'\rangle = \langle\bar\alpha, v\rangle + \langle\beta, \bar u\rangle\;.$$

In translating this into our setting the main problem is γ^a and $\not\nabla$, if we wish to allow for a possibly degenerate vierbein. But note that we have (§2.4)

$$\tilde{\not\nabla}\psi := \tilde\gamma^{\#}\,\nabla\psi : \boldsymbol{M} \to \tilde{\boldsymbol{H}}\otimes T^*\boldsymbol{M}\otimes\tilde{\boldsymbol{W}}\;,$$

so that

$$\langle\bar\psi, \tilde{\not\nabla}\psi\rangle : \boldsymbol{M} \to \boldsymbol{L}^{-3}\otimes\tilde{\boldsymbol{H}}\otimes T^*\boldsymbol{M} = \boldsymbol{L}^{-4}\otimes\boldsymbol{H}\otimes T^*\boldsymbol{M}\;.$$

Hence

$$\tilde\eta\Big(\langle\bar\psi, \tilde{\not\nabla}\psi\rangle\wedge\theta\wedge\theta\wedge\theta\Big) : \boldsymbol{M} \to \mathbb{C}\otimes\wedge^4 T^*\boldsymbol{M}\;.$$

Moreover let $m : \boldsymbol{M} \to \boldsymbol{L}^{-1}$ be a constant section. Then

$$\tilde\eta\Big(m\,\langle\bar\psi\,,\psi\rangle\,\theta\wedge\theta\wedge\theta\wedge\theta\Big) : \boldsymbol{M} \to \wedge^4 T^*\boldsymbol{M}\;.$$

Collecting everything, we obtain the Lagrangian $\mathcal{L}_{\mathrm{D}} : \boldsymbol{M} \to \wedge^4 T^*\boldsymbol{M}$ given by

$$\mathcal{L}_{\mathrm{D}} = \tilde\eta\,\Big[\Big(\tfrac{i}{2}\,(\langle\bar\psi, \tilde{\not\nabla}\psi\rangle - \langle\overline{\tilde{\not\nabla}\psi}, \psi\rangle) - m\,\langle\bar\psi\,,\psi\rangle\,\theta\Big)\wedge\theta\wedge\theta\wedge\theta\Big]\;.$$

Note how we replaced the combination $mc/$˜ with m only, which is essentially the usual setting ˜ $= c = 1$. Actually, in the present context, the particle's mass and the Planck constant appear only in this combination.

In 2-spinor notation we have

$$\mathcal{L}_{\mathrm{D}} = \tilde\eta\,\Big[\Big(\tfrac{i}{\sqrt2}\,(\nabla u\otimes\bar u - u\otimes\nabla u + \tilde g^{\#}(\bar\alpha\otimes\nabla\alpha - \nabla\bar\alpha\otimes\alpha))$$
$$- m\,(\langle\alpha,\bar u\rangle + \langle\bar\alpha, u\rangle)\,\theta\Big)\wedge\theta\wedge\theta\wedge\theta\Big]\;,$$

with coordinate expression

$$\ell_{\mathrm{D}} = \varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\Big[\Big(\tfrac{i}{\sqrt2}\,(\nabla_a u^A\,\bar u^{B^\cdot} - u^A\,\nabla_a\bar u^{B^\cdot} + \varepsilon^{CA}\bar\varepsilon^{D^\cdot B^\cdot}(\bar\alpha_C\,\nabla_a\alpha_{D^\cdot} - \nabla_a\bar\alpha_C\,\alpha_{D^\cdot}))\,t^{\lambda}{}_{AB^\cdot}$$
$$- m\,(\bar\alpha_A u^A + \alpha_{A^\cdot}\bar u^{A^\cdot})\,\theta^\lambda_a\Big)\theta^\mu_b\theta^\nu_c\theta^\rho_d\Big]\;.$$

Next we compute the Euler-Lagrange operator $\mathcal{E}_{\mathrm{D}}$. The $\bar u$-component is

$$(\mathcal{E}_{\mathrm{D}})_{B^\cdot} = \frac{\partial\ell_{\mathrm{D}}}{\partial\bar u^{B^\cdot}} - \partial_a u^A\,\frac{\partial}{\partial u^A}\frac{\partial\ell_{\mathrm{D}}}{\partial\bar u^{B^\cdot}_a} - \partial_a\theta^\mu_b\,\frac{\partial}{\partial\theta^\mu_b}\frac{\partial\ell_{\mathrm{D}}}{\partial\bar u^{B^\cdot}_a} =$$
$$= \varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\Big(\sqrt2\,i\,\nabla_a u^A\,t^{\lambda}{}_{AB^\cdot} - m\,\alpha_{B^\cdot}\theta^\lambda_a\Big)\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d + \mathcal{B}_{B^\cdot}\;,$$

where

$$\mathcal{B}_{B^\cdot} = \tfrac{i}{\sqrt{2}} \varepsilon^{abcd} \varepsilon_{\lambda\mu\nu\rho}\, t^{\chi}{}_{AB^\cdot} u^A \left(3\,\delta^\lambda_\chi\, \partial_a \theta^\mu_b + \tfrac{1}{4}\Gamma_a{}^\beta{}_\gamma (\bar\Sigma_\beta{}^{\gamma\pi}\Sigma_\pi{}^\lambda{}_\chi + \Sigma_\beta{}^{\gamma\pi}\bar\Sigma_\pi{}^\lambda{}_\chi)\,\theta^\mu_b\right)\theta^\nu_c\,\theta^\rho_d\,.$$

The $\bar\alpha$-component is

$$\begin{aligned}(\mathcal{E}_{\rm D})^A &= \frac{\partial \ell_{\rm D}}{\partial \bar\alpha_A} - \partial_a \alpha_{B^\cdot} \frac{\partial}{\partial \alpha_{B^\cdot}} \frac{\partial \ell_{\rm D}}{\partial \bar\alpha_{A,a}} - \partial_a \theta^\mu_b \frac{\partial}{\partial \theta^\mu_b}\frac{\partial \ell_{\rm D}}{\partial \bar\alpha_{A,a}} = \\ &= \varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\left(\sqrt{2}\, i\,\eta^{\lambda\kappa}\tau_\kappa{}^{AB^\cdot}\nabla_a \alpha_{B^\cdot} - m\, u^A\,\theta^\lambda_a\right)\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d + \mathcal{B}^A\,,\end{aligned}$$

where

$$\mathcal{B}^A = \tfrac{i}{\sqrt{2}} \varepsilon^{abcd} \varepsilon_{\lambda\mu\nu\rho}\, \tau_\chi{}^{AB^\cdot} \alpha_{B^\cdot} \left(3\,\eta^{\lambda\chi}\, \partial_a \theta^\mu_b - \tfrac{1}{4}\eta^{\lambda\kappa}\Gamma_a{}^\beta{}_\gamma (\bar\Sigma_\beta{}^{\gamma\pi}\bar\Sigma_{\pi\kappa}{}^\chi + \Sigma_\beta{}^{\gamma\pi}\Sigma_{\pi\kappa}{}^\chi)\,\theta^\mu_b\right)\theta^\nu_c\,\theta^\rho_d\,.$$

The Γ-component is

$$\begin{aligned}\frac{\partial \ell_{\rm D}}{\partial \Gamma_a{}^\beta{}_\gamma} = \tfrac{i}{4\sqrt{2}}\,\varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\Big[&(\bar\Sigma_\beta{}^{\gamma\pi}\Sigma_\pi{}^\lambda{}_\chi - \Sigma_\beta{}^{\gamma\pi}\bar\Sigma_\pi{}^\lambda{}_\chi) t^\chi{}_{AB^\cdot} u^A\,\bar u^{B^\cdot} + \\ &+ \eta^{\lambda\kappa}\,(\bar\Sigma_\beta{}^{\gamma\pi}\bar\Sigma_{\pi\kappa}{}^\chi - \Sigma_\beta{}^{\gamma\pi}\Sigma_{\pi\kappa}{}^\chi)\,\tau_\chi{}^{CD^\cdot}\bar\alpha_C\,\alpha_{D^\cdot}\Big]\,\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d\,.\end{aligned}$$

The θ-component is

$$\begin{aligned}(\mathcal{E}_{\rm D})^c_\nu = \frac{\partial \ell_{\rm D}}{\partial \theta^\nu_c} = \varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\Big[\Big(&3\,\tfrac{i}{\sqrt{2}}\,(\nabla_a u^A\,\bar u^{B^\cdot} - u^A\,\nabla_a \bar u^{B^\cdot} + \varepsilon^{CA}\bar\varepsilon^{D^\cdot B^\cdot}(\bar\alpha_C\,\nabla_a\alpha_{D^\cdot} - \nabla_a\bar\alpha_C\,\alpha_{D^\cdot}))\,t^\lambda{}_{AB^\cdot} \\ &- 4\,m\,(\bar\alpha_A\,u^A + \alpha_{A^\cdot}\bar u^{A^\cdot})\,\theta^\lambda_a\Big)\theta^\mu_b\,\theta^\rho_d\Big]\,.\end{aligned}$$

The A-component is

$$\begin{aligned}(\mathcal{E}_{\rm D})^a &= \frac{\partial \ell_{\rm D}}{\partial A_a} = \\ &= \sqrt{2}\,\varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\left(u^A\,\bar u^{B^\cdot} + \varepsilon^{CA}\,\bar\varepsilon^{D^\cdot B^\cdot}\,\bar\alpha_C\,\alpha_{D^\cdot}\right)t^\lambda{}_{AB^\cdot}\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d = \\ &= \sqrt{2}\,\varepsilon^{abcd}\varepsilon_{\lambda\mu\nu\rho}\left(t^\lambda{}_{AB^\cdot}u^A\,\bar u^{B^\cdot} + \eta^{\lambda\mu}\,\tau_\mu{}^{CD^\cdot}\,\bar\alpha_C\,\alpha_{D^\cdot}\right)\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d = \\ &= \sqrt{2}\,\varepsilon_{\lambda\mu\nu\rho}\left((u\otimes\bar u)^\lambda + \eta^{\lambda\beta}(\bar\alpha\otimes\alpha)_\beta\right)\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d = \\ &= \varepsilon_{\lambda\mu\nu\rho}\,\bar\psi\tilde\gamma^\lambda\psi\,\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d\,.\end{aligned}$$

Finally, as anticipated, the G-component of $\mathcal{E}_{\rm D}$ vanishes identically:

$$\frac{\partial \ell_{\rm D}}{\partial G_a} = 0\,.$$

5.5 Electromagnetic Lagrangian

Unfortunately we are not able to write, in terms of our fields, a Lagrangian density which is not singular in the degenerate case and yields something resembling the usual Maxwell equations in the non-degenerate case. Then we simply translate the usual electromagnetic Lagrangian in our formalism:

$$\begin{aligned}\mathcal{L}_{\rm em} &= \langle g^\# \otimes g^\#, F\otimes F\rangle\,\eta = g^{ac}\,g^{bd}\,F_{ab}\,F_{cd}\,\eta = \\ &= \eta^{\lambda\nu}\,\eta^{\mu\rho}\,(\theta^{-1})^a_\lambda\,(\theta^{-1})^b_\mu\,(\theta^{-1})^c_\nu\,(\theta^{-1})^d_\rho\,F_{ab}\,F_{cd}\,\det(\theta)\,\xi\,,\end{aligned}$$

with $F := 2\,dA$. This Lagrangian is already dimensionless, namely it is valued in $\wedge^4 T^*\boldsymbol{M}$, and contributes to the A- and θ-components of the Euler-Lagrange operator.

For semplicity we use the shorthand

$$\breve{\theta}^{abcd} := \eta^{\lambda\nu}\,\eta^{\mu\rho}\,(\theta^{-1})^a_\lambda\,(\theta^{-1})^b_\mu\,(\theta^{-1})^c_\nu\,(\theta^{-1})^d_\rho\,\det(\theta)\ ,$$
$$\breve{\theta}^{abcd}{}_k := \partial_k\theta^X_q\,\frac{\partial}{\partial\theta^X_q}\,\breve{\theta}^{abcd}\ ,$$

so that

$$\ell_{\mathrm{em}} = F_{ab}\,F_{cd}\,\breve{\theta}^{abcd}\ .$$

The A-component of $\mathcal{E}_{\mathrm{em}}$ is

$$\begin{aligned}(\mathcal{E}_{\mathrm{em}})^a &= -\partial_b F_{cd}\frac{\partial}{F_{cd}}\frac{\partial\ell_{\mathrm{em}}}{\partial A_{a,b}} - \partial_b\theta^X_q\frac{\partial}{\theta^X_q}\frac{\partial\ell_{\mathrm{em}}}{\partial A_{a,b}} = \\ &= 2\left[(\breve{\theta}^{abcd}{}_b - \breve{\theta}^{bacd}{}_b)F_{cd} + (\breve{\theta}^{abcd} - \breve{\theta}^{bacd})\partial_b F_{cd}\right] = \\ &= (d * F)_{bcd}\,\varepsilon^{abcd}\ ,\end{aligned}$$

where $*F$ is the Hodge-dual of F, that is, in coordinates,

$$(*F)_{hj} = \breve{\theta}^{abcd}\,F_{ab}\,\varepsilon_{cdhj}\ .$$

The θ-component of $\mathcal{E}_{\mathrm{em}}$ is

$$(\mathcal{E}_{\mathrm{em}})^c_\nu = F_{ab}\,F_{hd}\,\frac{\partial}{\partial\theta^\nu_c}\breve{\theta}^{abhd}\ .$$

5.6 Gauge symmetry

Having gauged away the conformal (dilaton) symmetry, the fundamental bundle in our approach is $\boldsymbol{U}\to\boldsymbol{M}$. Since any frame of $\boldsymbol{S}$ yields a normalized frame of $\boldsymbol{U}$ (§2.4), we have on $\boldsymbol{U}$ the distinguished family of normalized frames. These constitute a principal bundle $\P\to\boldsymbol{M}$, whose structure group is constituted by matrices of the form $|\det K|^{-1/2}\,(K)$, with $(K)\in Gl(2,\mathbb{C})$. Namely the structure group is constituted by all complex 2×2 matrices having the module of the determinant equal to 1. This group is

$$\begin{aligned}\widetilde{Sl}(2,\mathbb{C}) := U(1)\tilde{\times}\,Sl(2,\mathbb{C}) &:= \big(U(1)\times Sl(2,\mathbb{C})\big)/\!\sim\, = \\ &= \big(U(1)\times Sl(2,\mathbb{C})\big)/\mathbb{Z}_2\ ,\end{aligned}$$

where $\sim$ is the equivalence relation $(x,X)\sim(x',X')\Leftrightarrow xX = x'X'$ and $\mathbb{Z}_2$ is identified with the normal subgroup of $U(1)\times Sl(2,\mathbb{C})$ generated by $(-1,-\mathbf{1})$. Equivalently, we can say that the structure of $\boldsymbol{U}$ is preserved by the group

$$\widetilde{Sl}(\boldsymbol{U}) = \widetilde{Sl}(\boldsymbol{S}) := U(1)\tilde{\times}\,Sl(\boldsymbol{S})\ .$$

Let moreover $\P_{\tilde{\boldsymbol{H}}}\to\boldsymbol{M}$ be the principal bundle of all positively oriented (§2.3) and future-oriented (§2.5) orthonormal frames of $\tilde{\boldsymbol{H}}\to\boldsymbol{M}$, with structure group the special orthochronous Lorentz group $L^{+\uparrow}$. The map which associates with a normalized frame of $\boldsymbol{U}$ the corresponding 'Pauli' frame of $\tilde{\boldsymbol{H}}$ is a principal bundle epimorphism.

Then we see that the group of automorphisms of the theory is $\widetilde{Sl}(\boldsymbol{S})\equiv\widetilde{Sl}(\boldsymbol{U})$.

The relation to the 4-spinor approach [CJ96] is readily found. If $K\in\widetilde{Sl}(\boldsymbol{S})$ then $K\oplus\bar{K}\in\mathrm{End}(\boldsymbol{W})$. Actually it turns out that $K\oplus\bar{K}\in Spin^{c\uparrow}(\boldsymbol{W})$, the time-orientation preserving component of the 'complexified' [BLM89] spin group $Spin^c(\boldsymbol{W}) := U(1)\tilde{\times}\, Spin(\boldsymbol{W})$, where $Spin(\boldsymbol{W}) \subset \mathrm{End}(\boldsymbol{W})$ is defined as usual in terms of the Dirac map $\tilde{\gamma}$. Moreover one finds that the map

$$\widetilde{Sl}(\boldsymbol{S}) \to Spin^{c\uparrow}(\boldsymbol{W}) : K \mapsto K \oplus \bar{K}$$

is a group isomorphism.

5.7 Field equations

We briefly discuss the field equations $\mathcal{E}[f] = 0$, where $\mathcal{E} = \mathcal{E}_{\mathrm{g}} + \mathcal{E}_{\mathrm{em}} + \mathcal{E}_{\mathrm{D}}$ and $f : \boldsymbol{M} \to \boldsymbol{E}$ denotes the whole of our fields.

The θ-component gives the Einstein equation

$$(\mathcal{E}_{\mathrm{g}})^c_\nu = -(\mathcal{E}_{\mathrm{em}} + \mathcal{E}_{\mathrm{D}})^c_\nu \;,$$

where, as we saw, the left-hand side is essentially, in the non-degenerate case, the Einstein tensor, while the right-hand side can be viewed as the sum of the energy-momentum tensors of the electromagnetic field and of the spinor field, respectively. In general, the electromagnetic term is singular in the degenerate case, while the other two terms are not singular.

The Γ-component gives the equation for torsion

$$(\mathcal{E}_{\mathrm{g}})^{a\,\beta}_{\;\alpha} = -(\mathcal{E}_{\mathrm{D}})^{a\,\beta}_{\;\alpha} \;.$$

From this we see that the spinor field is a source for torsion, and that in this context we cannot formulate a torsion-free theory. Note also that this equation is non-singular in the degenerate case.

The A-component gives the second Maxwell equation

$$\varepsilon^{abcd}[(d * F)_{bcd} + \mathcal{J}_{bcd})] = 0 \;,$$

where

$$\mathcal{J}_{bcd} := \sqrt{2}\,\varepsilon_{\lambda\mu\nu\rho}\Big((u\otimes\bar{u})^\lambda + \eta^{\lambda\beta}(\bar{\alpha}\otimes\alpha)_\beta\Big)\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d = \varepsilon_{\lambda\mu\nu\rho}\,\bar{\psi}\tilde{\gamma}^\lambda\psi\,\theta^\mu_b\,\theta^\nu_c\,\theta^\rho_d$$

is the current. This can be written in the usual form (we are in the non-degenerate case)

$$\delta F := -*d*F = \bar{\psi}\gamma^{\#}\psi \;.$$

The $\bar{\psi}$-component, namely the $\bar{u}$- and $\bar{\alpha}$-components, are $(\mathcal{E}_{\mathrm{D}})_A = 0$ and $(\mathcal{E}_{\mathrm{D}})^{B'} = 0$ (while the ψ-component gives the conjugate equation). These give a modified (non-linear) form of the Dirac equation. The modification is constituted by the terms $\mathcal{B}_{B'}$ and $\mathcal{B}^A$. In fact we can express $\partial_b\theta^\nu_c$ algebraically in terms of Γ and ψ via the torsion equation, and replace for this derivative in $\mathcal{B}_{B'}$ and $\mathcal{B}^A$. This kind of torsion-related non-linearity is known to arise in the context of Einstein-Cartan-Dirac fields [HCMN95, GH96]. Its possible physical meaning is an argument of discussion.

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