BORN'S RECIPROCITY IN THE CONFORMAL DOMAIN

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Abstract. Max Born's reciprocity principle is revisited and complex four dimensional Kähler manifold $D_4 \approx SU(2,2)/S(U(2) \times U(2))$ is proposed as a replacement for space-time on the micro scale. It is suggested that the geodesic distance in D_4 plays a role of a quark binding super-Hamiltonian.

1. Introduction

Some 55 years ago, in the Scottish city of Edinburgh, Max Born wrote '*A suggestion for unifying quantum theory and relativity*'[Born, 1938], the paper that introduced his 'principle of reciprocity'. He started there with these words:

> 'There seems to be a general conviction that the difficulties of our present theory of ultimate particles and nuclear phenomena (the infinite values of the self energy, the zero energy and other quantities) are connected with the problem of merging quantum theory and relativity into a consistent unit. Eddington's book, "Relativity of the Proton and the Electron", is an expression of this tendency; but his attempt to link the properties of the smallest particles to those of the whole universe contradicts strongly my physical intuition. Therefore I have considered the question whether there may exist (other possibilities of unifying quantum theory and the principle of general invariance, which seems to me the essential thing, as gravitation by its order of magnitude is a molar effect and applies only to masses in bulk, not to the ultimate particles. I present here an idea which seems to be attractive by its simplicity and may lead to a satisfactory theory. '

Born then went on to introduce the *principle of reciprocity* - a primary symmetry between coordinates and momenta. He explained that

> 'The word *reciprocity* is chosen because it is already generally used in the lattice theory of crystals where the motion of the particle is described in the p-space with help of the *reciprocal lattice*.'

A year later, in a paper "*Reciprocity and the Number 137. Part I*", [Born, 1939] he makes an attempt to derive from his new principle the numerical

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value of the fine structure constant. [1] The most recent and clear exposition of the principle of reciprocity appears in his paper '*Reciprocity Theory of Elementary Particles* ', published in 1949 in honor of 70th birthday of Albert Einstein [Born, 1949]. The following extensive quotation from the Introduction to this paper brings us closer to Born's original motivations.

> 'The theory of elementary particles which I propose in the following pages is based on the current concepts of quantum mechanics and differs widely from the ideas which Einstein himself has developed in regard to this problem.(...) Relativity postulates that all laws of nature are invariant with respect to such linear transformations of space time $x^k = (\mathbf{x}, t)$ for which the quadratic form $R = x^k x_k = t^2 - \mathbf{x}^2$ is invariant (...). The underlying physical assumption is that the 4-dimensional distance $r = R^{\frac{1}{2}}$ has an absolute significance and can be measured. This is natural and plausible assumption as long as one has to do with macroscopic dimensions when measuring rods and clocks can be applied. But is it still plausible in the domain of atomic phenomena? (...) I think that the assumptions of the observability of the 4-dimensional distance of two events inside atomic dimensions is an extrapolation which can only be justified by its consequences; and I am inclined to interpret the difficulties which quantum mechanics encounters in describing elementary particles and their interactions as indicating the failure of this assumption.
>
> The well-known limits of observability set by Heisenberg's uncertainty rules have little to do with this question; they refer to the measurements and momenta of a particle by an instrument which defines a macroscopic frame of reference, and they can be intuitively understood by taking into account that even macroscopic instruments must react according to quantum laws if they are of any use for measuring atomic phenomena. Bohr has illustrated this by many instructive examples. The determination of the distance $R^{\frac{1}{2}}$ of two events needs two neighboring space-time measurements; how could they be made with macroscopic instruments if the distance is of atomic size?
>
> If one looks at this question from the standpoint of momenta, one encounters another paradoxical situation. There is of course a quantity analogous to R, namely $P = p^2 = p_k p^k = E^2 - \mathbf{p}^2$, where $p_k = (\mathbf{p}, E)$ represents momentum and energy. But this is not a continuous variable as it represents the square of the rest mass. A determination of P means therefore not a real measurement but a choice between a number of values corresponding to the particles with which one has possibly to do. (...) It looks therefore, as if the distance P in momentum space is capable of an infinite number of discrete values which can be roughly determined while the distance R in coordinate space is not an observable quantity at all.
>
> This lack of symmetry seems to me very strange and rather improbable. There is strong formal evidence for the hypothesis , which I have called *the principle of reciprocity*, that the laws of nature are symmetrical with regard to space-time and momentum-energy, or more precisely, that they are invariant

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[1] He failed, but many years later Armand Wyler [Wyler, 1968,1969,1971] obtained a reasonable value by playing, as we shall see, with a similar geometrical idea. Wyler failed however in another respect: he was unable to formulate all the principles that are necessary to justify his derivation . His work was criticized (cf. [Robertson, 1971; Gilmore, 1971; Vigier, 1976]), his ideas not understood, his name disappeared from the lists of publishing scientists.

under the transformation

$$x_k \to p_k, \qquad p_k \to -x_k. \tag{I.1}$$

The most obvious indications are these. The canonical equations of classical mechanics

$$\dot{x}^k = \partial H/\partial p_k, \qquad \dot{p}_k = -\partial H/\partial x^k \tag{I.2}$$

are indeed invariant under the transformation (I.1), if only the first 3 components of the 4–vectors x^k and p_k are considered. These equations hold also in the matrix or operator form of quantum mechanics. The commutation rules

$$x^k p_l - p_l x^k = i\hbar\delta_l^k, \tag{I.3}$$

and the components of the angular momentum,

$$m_{kl} = x_k p_l - x_l p_k, \tag{I.4}$$

show the same invariance, for all 4 components. These examples are, in my opinion, strongly suggestive, and I have tried for years to reformulate the fundamental laws of physics in such a way that the reciprocity transformation (I.1) is valid (...). I found very little resonance in this endeavor; apart from my collaborators, K. Fuchs and K. Sarginson, the only physicist who took it seriously and tried to help us was A. Landé (...). But our efforts led to no practical results; there is no obvious symmetry between coordinate and momentum space, and one had to wait until new experimental discoveries and their theoretical interpretation would provide a clue. (...) There must be a general principle to determine all possible field equations, in particular all possible rest masses.(...) I shall show that the principle of reciprocity provides a solution to this new problem – whether it is the correct solution remains to be seen by working out all consequences. But the simple results which we have obtained so far are definitely encouraging (...).'

[2] The very problem of a serious contradiction between quantum theory and relativity was addressed again, in 1957, by E.P. Wigner in a remarkable paper '*Relativistic Invariance and Quantum Phenomena*', [Wigner, 1957]. Wigner starts with the assertion that 'there is hardly any common ground between the general theory of relativity and quantum mechanics'. He then goes on to analyze the limits imposed on space-time localization of events by quantum theory to conclude that:

'The events of the general relativity are coincidences, that is collisions between particles. The founder of the theory, when he created this concept, had evidently macroscopic bodies in mind. Coincidences, that is, collisions between such bodies, are immediately observable. This is not

[2] It must be said that later on, in his autobiographical book '*My life and my views*', [Born, 1968], Born hardly devoted more than a few lines to the principle of reciprocity. Apparently he was discouraged by its lack of success in predicting new experimental facts.

the case for elementary particles; a collision between these is something much more evanescent. In fact, the point of a collision between two elementary particles can be closely localized in space-time only in case of high–energy collisions.'

3

Wigner analyzes the quantum limitations on the accuracy of clocks, and he finds that "a clock, with a running time of a day and an accuracy of 10^{-8} second, must weigh almost a gram–for reasons stemming solely from uncertainty principles and similar considerations". [4]

2. Reciprocity the Twistor Way

Max Born's original idea of reciprocity was clear but imprecise. We will try to interpret it using more modern concepts. The interpretation below is ours. And so are its faults.

2.1. Interpretation

We will interpret the reciprocity symmetry (I.1) as a tangent space symmetry rather than as a global one. So, we assume that the fundamental arena D in which relativistic quantum processes take place is an 8-dimensional manifold with local coordinates (x^μ, p^μ). The symmetry (I.1) should hold in each tangent space. Since the square of the operation (I.1) is $-I$, we interpret (I.1) as the requirement that D should be equipped with a *complex structure*, which is respected by the fundamental equations. It is clear from Born's papers that D should be also endowed with a metric tensor. The simplest complex Riemannian manifolds are those that are Kählerian symmetric domains. I choose the Cartan domain $D_4 \approx SU(2,2)/S(U(2) \times U(2)) \approx SO(4,2)/S(O(2) \times O(2))$ as *the* candidate. It has many nice properties - some of them will be discussed later. There are also many possible objections against such a choice. Let me try to anticipate some of them.

– $D4$ has *positive-definite* metric - it cannot contain Minkowski space

= True, indeed. On the other hand one can argue that according to Born's original idea, and according to the analysis by Wigner, Minkowski space-time of events is only an *approximation*. High–energy or high–mass approximation. Thus it is reassuring that the *Shilov*

[3] I will return to this conclusion when interpreting space-time as the Shilov boundary of the conformal domain D_4.

[4] In 1986 Károlyházy et al. in the paper 'On the possible role of gravity in the reduction of the wave function', [Károlyházy, 1986], presented another analysis of the imprecision in space–time structure imposed by the quantum phenomena. They proposed 'to put the *proper amount of haziness* into the space–time structure'. Their ideas, as well as the ideas of a "stochastic space-time" most notably represented by E. Prugovečky (cf. [Prugovečki, 1991]) and references therein) all point in a similar direction.

boundary of D_4 (the important concept that will be discussed later) is naturally isomorphic to the (compactified) Minkowski space endowed with its *indefinite conformal structure*. Let us interpret the points of $D4$ as elementary micro *event–processes*, that is micro-events accompanied by energy transfers. A coordinate of such an event is $z^k = x^k + \hbar p^k/p^2$, with $p^2 = E^2 - \mathbf{p}^2 > 0$ (see Sec. ???). In the limit of large energy transfers $\hbar^2/p^2 \to 0$ the positive definite metric blows up. What remains is the Minkowskian conformal metric for $z^k = x^k + 0$ - the finite part of the Shilov boundary. The positive definiteness of the Riemannian metric on D_4 can be thus viewed as an advantage rather than as a fault.

– D_4 is not invariant under time inversion.

= Indeed, time inversion is not a symmetry of D_4 – it would change the complex structure into the opposite one. We will see that when realizing D_4 as a part of the Grassmannian in $\mathbf{C}^4$ one gets automatically two copies of the domain. Then time inversion can be thought of as the transposition of these two copies. We consider D_4 as useful for the modeling of physical processes on a micro–scale (say, inside mesons and hadrons). We know that on this scale time–inversion need not be a symmetry. On the other hand such a primary arrow-of-time on a micro-scale may well be connected with the observed macroscopic irreversibility as dealt with in thermodynamics. Thus breaking of the time-inversion symmetry can also viewed as an advantage rather than as a fault.

– D_4 has *constant* curvature and it is hard to imagine how models based on D_4 can be constructed that include gravity and/or gauge fields.

= True, one of the original reasons for introducing the principle of reciprocity was unification of gravity and quantum theory. On the other hand, let us recall that, according to Born, gravitation 'is a molar effect and applies only to masses in bulk, not to the ultimate particles.'If so, and according to our interpretation above, there is no place for gravitation (and for other gauge fields as well) inside a meson or a hadron. Of, course, one could object that then there is also no place for space,time,energy and for momentum. It is of course an extrapolation, perhaps unjustified, that these concepts apply to such a micro-scale. However, extrapolating Einstein's scheme of general relativity into this domain would be unjustified even more. Therefore the idea that the primary arena of elementary event-processes is *homogenous* under a sufficient "zoom" may be rather attractive than appalling. [5]

– There is nothing new in the idea. Everything has been already said.

[5] I heard this idea from Rudolph Haag.

= This objection is a serious one. There are extensive papers dealing with the domain D_4, mainly by Roger Penrose and his group (cf. [Penrose and Rindler, 1986] and references there), but also by Odzijewicz and collaborators (see [Odzijewicz, 1976; Karpio et al., 1986; Odzijewicz, 1988] and references there), and by Unterberger [Unterberger, 1987]. Many of these papers are too difficult for me to understand all their conclusions. Therefore there is a chance that the ideas presented here are simplistic and naive, mainly owing to my inadequate knowledge. If so, I will beg your pardon, and I will do my best to (at least) present those ideas that, I believe, deserve propagation. [6]

3. Algebraic description of the conformal domain D_4

There are many ways of describing the same domain D_4. I choose the algebraic description because it is simple. On the other hand it so happens that many years ago I studied its geometry, by algebraic means, without being fully aware of the full impact of the study [Jadczyk, 1971].

Let V be a complex vector space of complex dimension $n = p+q$, equipped with a Hermitean scalar product $\langle\ ,\ \rangle$ of signature (p,q). The domain D_n^+ is then defined as the manifold of p dimensional, *positive* linear subspaces of V[7]. In the following we will write D_n to denote D_n^+. Let $L(V)$ denote the algebra of linear operators on V. For each subspace $W \in D_n$ let E_W denote the orthogonal projection on W, and let $S_W \equiv 2E_W - I$. Then $S_W = S_W^\star$, $S_W^2 = I$, and $(v,w)_{S_W} \doteq (v, S_W w)$ is a *positive definite* scalar product on V. The last statement follows from the fact that S_W reverses the sign on $W^\perp$. Conversely, if $S \in L(V)$ satisfies the three conditions above, then the subspace $W \doteq \{v : Sv = v\}$ is in D_n and $S = S_W$. Geometrically, S_W plays the role of a *geodesic reflection symmetry* with respect to the point $W \in D_n$. The parametrization of the points of D_n through their symmetries is in many respects the most convenient one - the fact that is little known! Whenever we speak about a point of D_n, we have in mind one of its representing objects: subspace W, projection E, or symmetry operator S. We will use the '$\star$'symbol to denote the Hermitean conjugate with respect to the indefinite scalar product on V. Given $S \in D_n$, the Hermitean conjugate of $Y \in L(V)$ with respect to the positive-definite scalar product $(u,v)_S$ will be denoted by Y^S. Notice that $Y^S = SY^\star S,\ \ Y^\star = SY^S$.

It is evident from the very definition that the unitary group $U(V)$ of $(V, \langle\ ,\ \rangle)$, which is isomorphic to $U(p,q)$, acts transitively on D_n with the

[6] A review with a different emphasis can also be found in [Coquereaux and Jadczyk, 1990]

[7] The orthocomplements of the subspaces from D_n^+ are q dimensional *negative* subspaces. They form D_n^-. For $p = q$ this is the second copy of D_n^+ – as mentioned in the discussion of time inversion above.

stability group $U(p) \times U(q)$. The same is true about $SU(p,q)$, which acts effectively on D_n, so that

$$D_n \simeq SU(p,q)/S(U(p) \times U(q))$$

. By differentiating the defining equations

$$S = S^{\star}, \qquad S^2 = I, \tag{1}$$

of D_n we find that the tangent space T_S at S can be identified with the set of operators $X \in L(V)$ such that

$$X = X^{\star}, \quad \text{and} \quad XS + SX = 0. \tag{2}$$

Suppose now that $p = q$, thus $n = 2p$ (the most symmetric case). Call a basis $\{e_i\}$ in V isotropic if the scalar product of V in this basis reads $\langle v, w\rangle = v^{\dagger}Gw$, where G is the block matrix

$$G = \begin{pmatrix} 0_p & iI_p \\ -iI_p & 0_p \end{pmatrix}. \tag{3}$$

Fix an isotropic basis, then D_n is isomorphic to the space of all $p \times p$ complex matrices T such that

$$i(T^{\star} - T) > 0, \tag{4}$$

the isomorphism $W \Longleftrightarrow T$ being given by

$$W = \{\begin{pmatrix} Tu \\ u \end{pmatrix} : \; u \in \mathbf{C}^p\}. \tag{5}$$

This parametrization defines complex structure on D_n. In terms of the operators X of Eq.(2) the complex structure J_S of the tangent space T_S at S is given by the map $J_S : X \rightarrow iXS$. Notice that (in the chosen isotropic basis) the orthogonal subspace to W_T is

$$W_T{}^{\perp} = \{\begin{pmatrix} T^{\dagger}u \\ u \end{pmatrix} : \; u \in \mathbf{C}^p\} = W_{T^{\dagger}}. \tag{6}$$

D_n is naturally equipped with an $U(V)$ – invariant positive definite Riemannian metric:

$$g(X,Y)_S \doteq -Tr(XY), \qquad X, Y \in T_S. \tag{7}$$

That g is positive definite follows from $X = X^{\star} = -X^S$, thus

$$g(X,X)_S = -Tr(XX) = Tr(X^S X) > 0 \quad \text{for} \quad X \neq 0. \tag{8}$$

D_n carries also an $U(V)$ – invariant symplectic structure ω: [8]

$$\omega(X,Y)_S \doteq g(X, J_S Y)_S = iTr(SXY). \tag{9}$$

D_n is a homogeneous Kählerian manifold. For $p = q = 2$ its interpretation as a conformal–relativistic phase space comes from the T–parametrization: [9] with T as in Eq. (5), we write

$$T = t^\mu \sigma_\mu = (x^\mu + \frac{q^\mu}{q^2})\sigma_\mu, \tag{10}$$

where $q^\mu = i\hbar p^\mu$, and $\sigma_\mu = \{I_2, \sigma\}$ are the Pauli matrices. The condition (4) reads now $p^2 = (p^0)^2 - \mathbf{p}^2 > 0$. Thus topologically, and also with respect to the action of the Poincaré group, D_4 is nothing but the future tube of the Minkowski space, endowed with a nontrivial Riemannian metric. It is to be stressed that special conformal transformations act on the variables p^μ not in the way one would normally expect. Thus (x^μ, p^μ) refer to some *extended process* rather than to a point event. Till now no interpretation of the points of D_4 in terms of space-time concepts, i.e. an interpretation that would explain their transformation properties, has been given.

The second important representation of D_n is as a bounded domain in $\mathbf{C}^{p^2}$. This representation can be obtained via the Cayley transform from the T-representation:

$$Z = i\frac{T-i}{T+i}, \qquad iT = \frac{Z+i}{Z-i}. \tag{11}$$

Geometrically, Z can be thought of as an orthogonal graph of the subspace W_T with respect to a fixed subspace $W_0 = W_{\{T=i\}}$. The condition (4) reads now $ZZ^\dagger < I$. The topological boundary ∂D_n is $(p^2 - 1)$ dimensional. The *Shilov* boundary $\check{\partial} D_n$ is defined as consisting of those points of ∂D_n at which functions analytic on the domain reach their maxima. $\check{\partial} D_n$ is isomorphic to the set of $p \times p$ unitary matrices; thus, for $p = 2$, to the compactified Minkowski space. $\check{\partial} D_n$ carries a unique $U(p,p)$–invariant conformal structure of signature (p,p). For $p = 2$ – the one induced by a flat Minkowski metric. The Cayley transform maps Minkowski space $t^\mu = x^\mu + 0$ onto the finite (affine) part of $\check{\partial} D_4$. We see from Eq. (10) that Minkowski space can interpreted as the *very–high–mass*, or *very–high–energy–momentum–transfer* limit of D_4. Elementary micro-processes that are characterized by very high energy-momentum transfers can be described as pure space-time events. It is only for such processes that the standard concepts of space, time and causality are applicable. For generic micro-processes there is no distinction between

[8] Although it is clear that ω is a non–degenerate, $U(V)$–invariant two-form, to prove that it is *closed* needs a computation.

[9] A justification for such a parametrization can be found in [Odzijewicz, 1976], [Coquereaux and Jadczyk, 1990]

space and time, no distinction between space-time and energy-momentum. This would be an extreme manifestation of the Born reciprocity idea! *Thus, we propose to consider* D_4 *as the replacement for space-time on the micro scale.* In an analogy to the harmonic oscillator, the (square of) geodesic distance in D_4 may play a role of the quark binding super-Hamiltonian. One obtains in this way, again in the spirit of Born's reciprocity, an interesting and non-trivial version of the relativistic harmonic oscillator. Here we can only sketch the idea. [10]

Given two points S, S' in D_n, the fundamental two-point object is the unitary operator $t(S', S) \doteq (S'S)^{\frac{1}{2}}$. Many of the algebraic properties of these operators (including the case of $n = \infty$) have been studied in [Jadczyk, 1971]. Notice that $t(S', S)$ is unitary w.r.t the indefinite scalar product of V, but positive w.r.t both p.d. scalar products $(u, v)_S$, $(u, v)_{S'}$. In the next paragraph we will show that the map

$$X \longmapsto t(S'S)Xt(S'S)^\star$$

is the geodesic transport from the tangent space at S to the tangent space at S'.

3.1. Reductive decomposition of $U(V)$

For the Lie algebra of $U(V)$ we have:

$$Lie(U(V)) = \{Y \in L(V) : Y = -Y^\star\}, \tag{12}$$

while $L(V)$ coincides with the complexified $Lie(U(V))$. The Killing form $B(X, Y)$ is then given by

$$B(X, Y) \approx Tr(XY). \tag{13}$$

Given $S \in D_n$, the isotropy subalgebra K_S at S is

$$K_S = \{X \in L(V) : X^\star = -X, [X, S] = 0\}. \tag{14}$$

Every $X \in L(V)$ can be uniquely decomposed as

$$X = X_S^+ + iX_S^-,$$

where

$$[X_S^\pm, S]_\mp = 0.$$

The decomposition is given by

$$X_S^+ = \frac{1}{2}(SXS + X),$$

[10] More can be found in the forthcoming Thesis of W. Mulak (cf. also [Mulak, 1992] for an SU(1,1) version)

$$X_S^- = \frac{i}{2}(SXS - X).$$

We have $(X^\star)_S^\pm = (X_S^\pm)^\star$, and also

$$Tr(X_S^+ Y_S^-) = 0, \qquad \forall X, Y \in L(V).$$

Therefore the orthogonal complement of K_S w.r.t. the Killing form $B(X,Y)$ is the subspace $M_S \subset Lie(U(V))$ given by

$$M_S = \{X \in L(V) : X^\star = -X, XS + SX = 0\}.$$

3.2. $t(S', S)$ AS THE GEODESIC PARALLEL TRANSPORT

We will show that $t(S', S)$ implements parallel transport from the tangent space at S to that at S', and also how it can be used for computing of the geodesic distance between the two points. First notice that each geodesic through S is generated by a unique element $X \in M_S$ as follows (cf. [Kobayashi, 1969], p.192):

$$t \longmapsto S(t) = e^{tX} S e^{-tX} = e^{2tX} S, \tag{15}$$

the last equality follows from $XS + SX = 0$. If $Y(t)$ is a parallel vector field along $S(t)$, then (because D_n is a symmetric space; see [Chavel, 1972], p.64)

$$Y(t) = S(t/2)S(0)Y(0)S(0)S(t/2), \tag{16}$$

which by (15) gives

$$Y(t) = e^{tX} Y(0) e^{-tX}. \tag{17}$$

On the other hand

$$t(S(t), S(0)) = (S(t)S)^{\frac{1}{2}} = (e^{2tX})^{\frac{1}{2}} = e^{tX}, \tag{18}$$

and so

$$Y(t) = t(S(t), S)Y(0)t(S(t), S)^{-1}, \tag{19}$$

which proves that $t(S(t)S)$ is the parallel transport operator. To find the geodesic distance formula, notice that $e^{2tX}S$ is a geodesic through S with the tangent vector field $\dot{S} = 2Xe^{2tX}S$ of length $-Tr(\dot{S}^2) = 4Tr(X^2)$. For $Tr(X^2) = \frac{1}{4}$, $S(t)$ is parametrized by its length. But, from Eq.(18), we have that $tX = \ln t(S(t)S)$, $t^2X^2 = \ln^2 t(S(t)S)$, thus

$$dist(S, S(t)) = t = 4Tr(\ln^2 t(S(t)S))), \tag{20}$$

or

$$dist(S, S') = Tr(\ln^2(SS')). \tag{21}$$

4. Conclusions: quantum conformal oscillator

The relativistic quark model based on the Lorentz-covariant harmonic oscillator has been considered by many authors (cf. [Kim and Noz, 1991], and references there). Extending Max Born's reciprocity principle we propose to investigate a similar model, but based on the geometry of D_4.

For simplicity let us consider the *spinless* two-body problem in D_4. Quantum states of the two-body system will be described by analytic functions[11] $\Psi(S, S')$ on $D_4 \times D_4$, integrable with respect to an appropriate invariant measure. We take for super–Hamiltonian H of the system the Toeplitz projection of $dist(S, S')^2$. One can prove that by introducing the 'center of mass 'coordinates, the problem reduces to a one body problem. The spectrum of H can be computed in terms of the coherent states on D_4 (cf. [Mulak, 1992]). Such a model is nonrealistic, as it does not take into account spin. To consider spinning quarks we have to take for a model Hilbert space the space of sections of an appropriate vector bundle. The most natural one is the holomorphic tautological bundle Q^+ that associates to each $S \in D_4$ the subspace $W_S = \{u \in V : Su = u\}$. This bundle is endowed with a natural Hermitean connection. The operators $t(S, S')$ provide a natural parallel transport also in this bundle. Using its natural connection a Dirac–like operator can be constructed on Q^+. Much work must still have been done in order to see if models constructed along these lines have anything to do with reality.

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